15-251: Great Theoretical Ideas In Computer Science

Recitation 2

False Counting

What is wrong with the following counting arguments?

1. How many rolls of 5 dice contain exactly two 3s assuming all dice are identical?

   > First choose 2 dice to be threes. Then there are 5 possibilities for the remaining three dice, so we get $\binom{5}{2} \cdot 3^5$.

2. 20 people are sitting at a round table. How many ways are there of choosing 3 people from them so that no two of the chosen are neighbors?

   > There are 20 ways of choosing the first person, 17 ways of choosing the second person, since he can't sit next to or on top of the first person, and there are 14 ways of choosing the third person. The order we choose these people however does not matter, so we divide by 3! and get $\frac{20 \cdot 17 \cdot 14}{3!}$.

Exercises

3. Show that:
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

4. Show that:
$$\binom{n}{2}\binom{m}{2} = \sum_{i=1}^{n}\sum_{j=1}^{m}(n-i)(m-j)$$

5. How many ordered pairs (A, B) exist such that $A, B \subseteq \{1, \ldots, n\}$ and $|A \cap B| = 1$?

6. Simplify
$$\sum_{k=0}^{n}\left[(2k-n)\binom{n}{k}\right]$$

7. How many ways can one divide $\{1, \ldots, n\}$ into three labelled sets such that for all $1 \leq i < n$, i and $i+1$ are in different sets?

8. There are 123 people at a party. Prove that there exist two people whose birthdays are separated by no more than two days.

9. There are 123 people at a party. Prove that there exist two people who know the same number of people at the party (note that knowing is symmetric; if person A knows person B, then person B knows person A).

10. Prove that the number of ways to partition n into at most k parts is the same as the number of ways to partition $n+k$ into exactly k parts.

11. How many ways can one divide $\{1, \ldots, 2n\}$ into n disjoint, two-element subsets, where the list of the n subsets is ordered?