

15-251: Great Theoretical Ideas In Computer Science

Recitation 2 Solutions

False Counting

1. How many rolls of 5 dice contain exactly two 3s assuming all dice are identical?

The argument given double-counts the remaining dice. For example rolling 4,5,6 and rolling 5,6,4 would be counted as two different rolls when it should not since the dice are identical. The correct answer is $\binom{7}{4}$.

2. 20 people are sitting at a round table. How many ways are there of choosing 3 people from them so that no two of the chosen are neighbors?

The argument given didn't consider the situation where the first and second person are almost neighbors, meaning there is only one empty space between them. When this happens, there are 15 possible places for the third person. The correct answer is $\boxed{800}$ ways.

Exercises

3. Show that:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Consider picking k balls from a row of n ; this can be done in $\binom{n}{k}$ ways. Also, this can be divided into two cases: one where the first ball is picked (leaving $\binom{n-1}{k-1}$ ways to pick the remaining balls) and one where the first ball is not picked (leaving $\binom{n-1}{k}$ ways to pick the remaining balls).

4. Show that:

$$\binom{n}{2} \binom{m}{2} = \sum_{i=1}^n \sum_{j=1}^m (n-i)(m-j)$$

The left side counts the number of rectangles that can be found in a grid with n horizontal lines and m vertical lines. The right side does the same; it picks the top horizontal line, then the left vertical line, then counts the number of ways to pick the other corner.

5. How many ordered pairs (A, B) exist such that $A, B \subseteq \{1, \dots, n\}$, and $|A \cap B| = 1$?

There are n ways to pick the shared element. After that, there are 3 choices for each of the $n-1$ remaining elements; each can be in A , in B , or in neither. Hence, there are a total of $\boxed{n \cdot 3^{n-1}}$ such pairs of subsets.

6. Simplify

$$\sum_{k=0}^n \left[(2k-n) \binom{n}{k} \right]$$

Note that $(2k-n) \binom{n}{k} + (2(n-k)-n) \binom{n}{n-k} = (2k-n) \binom{n}{k} + (n-2k) \binom{n}{k} = 0$. Hence, pairing terms (and noticing that when n is even, the middle, unpaired term is also zero), yields a sum of $\boxed{0}$.

7. How many ways can one divide $\{1, \dots, n\}$ into three labelled sets such that for all $1 \leq i < n$, i and $i + 1$ are in different sets?

Let this quantity be $f(n)$. Then for any division of $f(n - 1)$, we can generate a unique division of n by adding n to one of the two parts that doesn't contain $n - 1$. Hence, $f(n) = 2f(n - 1)$. There are 3 options for the base case, $n = 1$. Hence, we have a recurrence for $f(n)$, and plugging in demonstrates that $f(n) = \boxed{3 \cdot 2^{n-1}}$ satisfies this recurrence.

8. There are 123 people at a party. Prove that there exist two people whose birthdays are separated by no more than two days.

Divide the year (of 366 days) into 122 blocks of 3 days; by the pigeon-hole principle, there must be two people whose birthdays fall into the same block.

9. There are 123 people at a party. Prove that there exist two people who know the same number of people at the party (note that knowing is symmetric; if person A knows person B , then person B knows person A).

There are two cases. If some person knows no one, then it is impossible for any other person to know everyone. On the other hand, if some person knows everyone, then it is impossible for another person to know no one. In either case, there are only 122 possibilities for the number of a people at the party a given person can know; the result follows by the pigeon-hole principle.

10. Prove that the number of ways to partition n into at most k parts is the same as the number of ways to partition $n + k$ into exactly k parts.

A bijection between partitions of n into at most k parts and partition of $n + k$ into exactly k parts is the function which adds enough 0s to the partition of n to have exactly k parts, then adds one to each term in the partition. This process is clearly reversible, and also clearly covers every possible partition.

11. How many ways can one divide $\{1, \dots, 2n\}$ into n disjoint, two-element subsets, where the list of the n subsets is ordered?

Consider all $(2n)!$ orderings of this set, with lines drawn every two elements. This clearly counts every partition, but we have overcounted by 2^n times, for the rearrangements of each two-element set. Hence, there are $\boxed{(2n)!/2^n}$ possible divisions.