15-251: Great Theoretical Ideas In Computer Science

Recitation 1

Failed Induction

In this course, you will find proof by induction to be an invaluable technique. For that reason, you should be careful to learn proper technique, or else you may end up 'proving' something that isn't true!
Explain what went wrong in the following proofs:

1. (a) **Claim:** $\sum_{i=0}^{n}(2i+1)3^i = n3^{n+1}$ for all natural numbers n.
Proof: Assume the claim is true for n, prove true for $n+1$.

$$\sum_{i=0}^{n+1}(2i+1)3^i = n3^{n+1} + (2n+3)3^{n+1} = (3n+3)3^{n+1} = (n+1)3^{n+2}$$

(b) **Claim:** $\log_3 n = \log_2 n$ for all natural numbers n.
Proof (by strong induction):
The inductive hypothesis is $P_n =$"$\log_3 n = \log_2 n$".
Base case: $\log_3 1 = 0 = \log_2 1$.
Inductive step: Assume that $\log_3 k = \log_2 k$ for all natural numbers $k \leq n$, and show it is true for $n+1$. Write $n+1$ as a product of two natural numbers p and q so that we have:

$$\log_3(n+1) = \log_3(pq) = \log_3 p + \log_3 q = \log_2 p + \log_2 q = \log_2(pq) = \log_2(n+1)$$

which is true by the inductive hypothesis.

(c) **Claim:** Every natural number is either prime or a perfect square.
Proof: As the inductive hypothesis, use $P_n =$"every number less than or equal to n is a prime or a perfect square."
P_1 is certainly true. Now consider n. If n is prime we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n-1$. By the inductive hypothesis, r and s are perfect squares, $r = u^2$ and $v = s^2$. Then $n = rs = u^2v^2 = (uv)^2$.

Simple Induction Proofs

2. Show that for every natural number $n \geq 5$, $2^n > n^2$.

3. Suppose a is a real number less than 0. Prove that for all natural number n, $a^n > 0$ if n is even and $a^n < 0$ if n is odd.

Planar Regions

4. Suppose you are allowed to draw n straight lines on the plane. Into how many regions can you divide the plane? Derive a recurrence and closed form.

5. Now suppose the rules have changed. Instead of drawing straight lines, you are allowed to draw V shapes. (Formally, a V shape consists of a point that is the endpoint of two rays, such that the two rays do not directly oppose each other). How many regions can you divide the plane into now? Again, find a recurrence and closed form.

Chess Board

6. Suppose you have an 8×8 chess board with two corner squares located at 1×1 and 8×8 missing, is it possible to fill the rest of the chess board with 1×2 dominos?

Game

7. Consider a row of n skittles. Each player takes turn knocking out any one or two consecutive skittles, if the player has no skittles to knock out, he loses. Who wins? (The answer may depend on n)