15-251: Great Theoretical Ideas In Computer Science

Recitation 1 Solutions

Failed Induction

In this course, you will find proof by induction to be an invaluable technique. For that reason, you should be careful to learn proper technique, or else you may end up 'proving' something that isn't true!
Explain what went wrong in the following proofs:

1. (a) **Claim:** $\sum_{i=0}^{n}(2i+1)3^i = n3^{n+1}$ for all natural numbers n.
Proof: Assume the claim is true for n, prove true for $n+1$.

$$\sum_{i=0}^{n+1}(2i+1)3^i = n3^{n+1} + (2n+3)3^{n+1} = (3n+3)3^{n+1} = (n+1)3^{n+2}$$

Solution: The proof is missing a base case. Since, this claim is true for none of the values, making the inductive step doesn't prove anything. The correct closed form is actually $n3^{n+1}+1$.

(b) **Claim:** $\log_3 n = \log_2 n$ for all natural numbers n.
Proof (by strong induction):
The inductive hypothesis is $P_n =$"$\log_3 n = \log_2 n$".
Base case: $\log_3 1 = 0 = \log_2 1$.
Inductive step: Assume that $\log_3 k = \log_2 k$ for all natural numbers $k \leq n$, and show it is true for $n+1$. Write $n+1$ as a product of two natural numbers p and q so that we have:

$$\log_3(n+1) = \log_3(pq) = \log_3 p + \log_3 q = \log_2 p + \log_2 q = \log_2(pq) = \log_2(n+1)$$

which is true by the inductive hypothesis.
Solution: The inductive step doesn't work when $n+1$ is prime, because in that case $n+1$ cannot be factored into two natural numbers less than $n+1$. So we cannot use our inductive hypothesis here.

(c) **Claim:** Every natural number is either prime or a perfect square.
Proof: As the inductive hypothesis, use $P_n =$"every number less than or equal to n is a prime or a perfect square."
P_1 is certainly true. Now consider n. If n is prime we are done. Otherwise, n can be factored as $n = rs$ with r and s less than or equal to $n-1$. By the inductive hypothesis, r and s are perfect squares, $r = u^2$ and $v = s^2$. Then $n = rs = u^2v^2 = (uv)^2$.
Solution: Here, the inductive step uses the inductive hypothesis incorrectly. We know that r and s are either perfect squares *or* primes. In this case, it wouldn't be possible to make an inductive hypothesis that works. However, in general you should make sure that your inductive hypothesis is specific and covers the cases you need for the inductive step.

Simple Induction Proofs

2. Show that for every natural number $n \geq 5$, $2^n > n^2$.

Solution:

- *Base Case:* when $n = 5$, $2^5 = 32 > 5^2 = 25$.
- *Inductive Hypothesis:* Assume that $2^n > n^2$ for some $n \geq 5$.

- *Inductive Step:* Show for $n+1$:

$$\begin{aligned} 2^{n+1} &= 2 \times 2^n \\ &> 2n^2 && \text{I.H.} \\ &= n^2 + n^2 \\ &> n^2 + 5n && \text{since } n \geq 5 \\ &> n^2 + 2n + 1 = (n+1)^2 \end{aligned}$$

3. Suppose a is a real number less than 0. Prove that for all natural number n, $a^n > 0$ if n is even and $a^n < 0$ if n is odd.

   Solution:

   - *Base Case:* if $n = 0$, $a^0 = 1 > 0$ and n is even.
   - *Inductive Hypothesis:* Assume for some $n \geq 0$, the claim is true.
   - *Inductive Step:* We show that the claim is true for $n+1$:
     If $n+1$ is even, than n is odd. By the inductive hypothesis, $a^n < 0$.
     Therefore, $a \times a^n > 0$ and $a^{n+1} > 0$.
     Otherwise, n is even. By the inductive hypothesis, $a^n > 0$.
     Therefore, $a \times a^n < 0$ and $a^{n+1} < 0$.

Planar Regions

4. Suppose you are allowed to draw n straight lines on the plane. Into how many regions can you divide the plane? Derive a recurrence and closed form.

   Solution: A plane with no lines is a single region. Consider adding a line to a plane with n lines, such that this new line is not parallel to any of the previous lines and does not intersect the line in any points of intersection of the original n lines. The new intersection points divide the new line into $n+1$ line segments, each of which creates a new region. Thus when the $n+1$st line is added, there is an increase of $n+1$ regions.

   Let $R(n)$ be the maximum number of regions in the plane created from drawing n lines. We obtain the following recurrence using the reasoning above:

$$R(n) = \begin{cases} 1 & n = 0 \\ R(n-1) + n & n > 0 \end{cases}$$

   We might realize that this $R(n)$ is just the sum of the first n integers, plus 1, thus leading us to propose that $R(n) = \frac{n(n+1)}{2} + 1$.
   Proof:

   - *Base case:* When $n = 0$, $R(0) = 1 = \frac{0(0+1)}{2} + 1$.
   - *Inductive Hypothesis:* Suppose for some $n \geq 0$, $R(n) = \frac{n(n+1)}{2} + 1$.
   - *Inductive Step:* We must show for $n+1$:

$$\begin{aligned} R(n+1) &= R(n) + (n+1) && \text{By recurrence for } R \\ &= \frac{n(n+1)}{2} + 1 + (n+1) && \text{I.H.} \\ &= \frac{n(n+1) + 2(n+1)}{2} + 1 \\ &= \frac{(n+1)(n+2)}{2} + 1 \end{aligned}$$

which is the form desired.

5. Now suppose the rules have changed. Instead of drawing straight lines, you are allowed to draw V shapes. (Formally, a V shape consists of a point that is the endpoint of two rays, such that the two rays do not directly oppose each other) How many regions can you divide the plane into now? Again, find a recurrence and closed form.

   Solution: Now, every time that we add a V, it is equivalent to adding 2 lines, but we lose 2 regions at the tip of the V. (We assume that we place the point of the V out so far in the plane that we still intersect all of the lines (i.e. halves of the V) in the plane.)

   So, our recurrence for $V(n)$ will be similar, but account for the two lost regions:

   $$V(n) = \begin{cases} 1 & n = 0 \\ V(n-1) + 2n + (2n-1) - 2 & n > 0 \end{cases}$$

   Here we might suppose that the recurrence has a quadratic form, $ax^2 + bx + c$:

   - Right away we know that $c = 1$, since $V(0) = 1$.
   - When $n = 1$, $V(1) = 1 + 2(1) + 1 - 2 = 2$, so $a + b + 1 = 2$, or, $a + b = 1$.
   - Also, when $n = 2$, we have $V(2) = 7$. This implies $4a + 2b + 1 = 7$. Then $4a + 2b = 6$, $4a + 2(1-a) = 6$, or $2a = 4$.
   - Thus we may conclude $a = 2, b = -1, c = 1$.

   Our proposed closed form is $V(n) = 2n^2 - n + 1$.
   Proof:

   - *Base case:* When $n = 0$, $V(0) = 1 = 2(0) - 0 + 1$.
   - *Inductive Hypothesis:* Suppose for some $n \geq 0$, $V(n) = 2n^2 - n + 1$.
   - *Inductive Step:* We must show for $n + 1$:

   $$\begin{aligned} V(n+1) &= V(n) + 2(n+1) + 2n - 2 && \text{By recurrence for } V \\ &= (2n^2 - n + 1) + 2(n+1) + (2n) - 2 && \text{I.H.} \\ &= 2n^2 + 3n + 1 \\ &= 2(n+1)^2 - 2n - 2 + 1 \\ &= 2(n+1)^2 - 2(n+1) + 1 \end{aligned}$$

   which matches the proposed closed form for $V(n+1)$.

Chess Board

6. Suppose you have an 8×8 chess board with two corner squares located at 1×1 and 8×8 missing, is it possible to fill the rest of the chess board with 1×2 dominos?

   Solution: It is impossible. No matter how you place the domino, the domino must cover one black square and one white square. However, both 1×1 and 8×8 are the same color. Therefore, there aren't equal amount of black and white squares on the board.

Game

7. Consider a row of n skittles. Each player takes turn knocking out any one or two consecutive skittles, if the player has no skittles to knock out, he loses. Who wins? (The answer may depend on n)

   Solution: First player always win regardless of how many skittles there are. If there are an even number of skittles, the first player can knock out the two center skittles and mirror the opponent's move after. If there are an odd number of skittles, the first player can mirror the opponent again after knocking out the one skittle in the center.