15-251: Great Theoretical Ideas In Computer Science

Homework 10 (due Tuesday, Nov 10)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

0. Warmup (0 points)

a. Show that any tree T has at least Δ leaves, where Δ is the maximum degree over all the vertices of T.

b. Give a linear-time algorithm that tests whether a given DFA rejects every string. *More to come*

1. Square Dancing (15 points)

There are n men and n women at a dance, at which n songs will be played. Prove that if after $r < n$ dances, no man has danced with the same woman twice, then it is possible for each man to dance with each woman exactly once.

2. Bipartite Graphs (15 points)

Show that, for any graph G, the following three properties are equivalent:

- G is bipartite
- G can be colored with just two colors.
- Every cycle in G has even length.

3. ytilibisiviD (20 points)

a. *(10 points)* Given a $b > 0$, construct a DFA over the language $\Sigma = \{0, 1\}$ that accepts a string $x_1, x_2, ..., x_k$ if and only if the binary number with digits $x_1 x_2 ... x_k$ is divisible by b. How many states does this machine have, as a function of b? For partial credit, do the specific case $b = 5$.

b. *(10 points)* Construct a DFA that accepts $x_1, x_2, ..., x_k$ if and only if the binary number $x_k, x_{k-1}, ..., x_1$ is divisible by b (i.e. the digits are given in reverse order). How many states does this machine have? For partial credit, do the specific case $b = 5$.

c. *(Extra credit: 10 points)* Generalize your construction from part b to an arbitrary number base system; given $n, b > 0$, construct a DFA over $\Sigma = \{0, ..., n-1\}$ that accepts a string $x_1, x_2, ..., x_k$ if and only if the base-n number $x_k x_{k-1} ... x_1$ is divisible by b. How many states does this machine have, as a function of b and n?

4. Weaving Words (15 points)

Let L_1 and L_2 be two regular languages over an alphabet Σ. Define the language

$Weave(L_1, L_2) = \{s_1 t_1 s_2 t_2 ... s_n t_n | s_1, ..., s_n, t_1, ..., t_n \in \Sigma, s_1 s_2 ... s_n \in L_1, t_1 t_2 ... t_n \in L_2\}$.

Prove that $Weave(L_1, L_2)$ is a regular language over Σ.

5. A Boring Problem (20 points)

Define a language L as *boring* if L is not regular, and no infinite subset of L is regular.

a. *(10 points)* Show that the language $\{a^n b^n\}$ over the alphabet $\{a, b\}$ is boring.

b. *(10 points)* A *palindrome* is a string w which is the same when read backward or forward; i.e. $w = w^{rev}$. Show that the language $\{w | w$ is a palindrome$\}$ is irregular, but not boring, over the alphabet $\{a, b\}$.

6. DFA DNA testing (15 points)

Give an $O(n^2)$ algorithm that determines whether two DFAs A and B are identical - that is, they accept exactly the same set of strings. Here, n is $max($number of states in A,number of states in $B)$.