15-251: Great Theoretical Ideas In Computer Science

Homework 10 Warmup Solutions

0. Warmup (0 points)

a. *Show that any tree T has at least Δ leaves, where Δ is the maximum degree over all the vertices of T.*

We know that any tree T with n vertices has $n-1$ edges. Also, summing the degrees ov every vertex counts every edge twice, so $\sum_{v \in V} degree(v) = 2(n-1)$. Now, assume there are at most $\Delta - 1$ leaves. Then summing the degrees gives at least $1 * \Delta + (\Delta - 1) * 1 + (n - \Delta) * 2$ - one vertex of degree delta, $\Delta - 1$ leaves, and the rest have degree at least two. This gives $(2n-1) > 2(n-1)$, which is a contradiction.

b. *Give a linear-time algorithm that tests whether a given DFA rejects every string.*

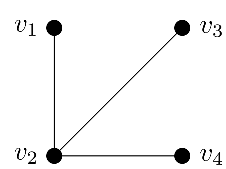
This is equivalent to testing whether or not there is an accept state that can be reached from the starting state. So, starting with q_0, build a list of all reachable states inductively as follows: at time t, if the list contains states $(q_0, ...q_i)$, add all states that have a transition from one of the nodes in the list.

Then, if you ever add an accept state, it must be reachable, so the DFA accepts some string. However, if at any step you fail to add any new nodes, you have enumerated all of the vertices reachable from the start state, so if this occurs and you have not yet added an accepting state, the DFA must reject every string.

c. *Compute*

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{1000000}$$

The matrix above is the adjacency matrix for the graph



and raising it to the 1000000^{th} power results in a matrix that gives the number of walks of length 1000000 between each pair of nodes.

There are several cases. First, take a vertex a other than v_2, and consider walks to a (potentially the same) vertex b that is also not v_2. The first edge in the walk is from a to v_2. The last edge in the walk is from v_2 to b. After the first edge, the walk may continue from v_2 to any one of the three other vertices, but the next edge after that must be back to v_2. This repeats until the last edge in the walk. There is thus a total of $(1000000 - 2)/2 = 499999$ choices to be made when selecting a walk, and 3 options at each choice. This means that there are 3^{499999} walks of length 1000000 from a to b.

Now, consider walks from v_2 to v_2. Starting with the first edge, every other edge must be from v_2 to another vertex, but after that, the next edge must be back to v_2. There are therefore $1000000/2 = 500000$ choices to be made, and 3 options at each choice. The total number of walks from v_2 to v_2 is 3^{500000}.

Finally, consider walks from v_2 to another vertex. Each such walk must have odd length. Because 1000000 is an even number, there are no walks of length 1000000 from v_2 to any other vertex. This means that

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{1000000} = \begin{bmatrix} 3^{499999} & 0 & 3^{499999} & 3^{499999} \\ 0 & 3^{500000} & 0 & 0 \\ 3^{499999} & 0 & 3^{499999} & 3^{499999} \\ 3^{499999} & 0 & 3^{499999} & 3^{499999} \end{bmatrix}$$