# 15-251: GTI Practice Test 2 Fall 2009

**Name:**

**AndrewID:**

**Section:**

**INSTRUCTIONS:**

- WRITE YOUR NAME, ANDREW ID AND SECTION IN THE BOX ABOVE.
- This is a closed book test. You may *not* use notes. You may *not* use a calculator.
- For the first **six** (6) problems, you do not need to give any reason or justification for your answers. However, partial credit may not be given unless you show your work.
- For the other problems you must give clear and complete proofs. In your proofs, you may quote and use any results presented during lectures; all other steps and results REQUIRE PROOFS. Please write clearly.
- This practice test does not have Reading Solutions and Extra Credit part. The real test will have these.

There is a total of 9 pages on this test. Make sure none are missing.

| Problem | Points | Score | Problem | Points | Score |
|---|---|---|---|---|---|
| 1 | 5 | | 8 | 15 | |
| 2 | 5 | | 9 | 15 | |
| 3 | 5 | | 10 | 25 | |
| 4 | 5 | | | | |
| 5 | 5 | | | | |
| 6 | 5 | | | | |
| 7 | 15 | | $\Sigma$ | 100 | |

## Repeat After Me.

This part is to test your ability to regurgitate basic facts. You should either have these facts memorized or be able to re-derive them on the spot.

**1**. **[5 points]** $\mathbb{I}$ is the set of irrational numbers. Which properties does the set $\mathbb{I} \cup \{0\}$ have under addition?

| Associativity | Closure | Identity | Inverse | Commutativity |
|---|---|---|---|---|
| | | | | |

**2**. **[5 points]** Take a *red* die and a *blue* die. Each die can come up either $1, 2, 3, 4, 5$, or $6$. How many ways are there to get

an even number on the red die **AND** an odd number on the blue die?

**3**. **[5 points]** How many numbers in $\mathbb{Z}_{35}$ have multiplicative inverses? (You **don't** have to list them.)

**4**. **[5 points]** Compute

$$39^{26} \pmod{35}$$

Give an integer answer in the range 0 to 34.

**5**. [**5 points**] Let $G$ be a group of prime order. Write down all the possible subgroups of $G$.

**6**. [**5 points**] Let $X$ be the random variable denoting the outcome of one roll of a standard 6-sided die. What is $\mathbb{E}[2X + 3]$?

**7**. **[10 points]** In the first 3 parts, let $X$ be a random variable; We assume **nothing** else about it. Mark the following statements as True or False. (Provide explanations if you must.)

T F $\mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X]$.

T F $\Pr(X \geq \mathbb{E}[X]) = \frac{1}{2}$.

T F $\Pr(X \geq 2\mathbb{E}[X]) \leq \frac{1}{2}$.

T F $\Pr(X = \mathbb{E}[X]) > 0$.

T F If $A$ and $B$ are independent events, then $\bar{A}$ and $\bar{B}$ are also independent events.

**Reading Solutions.**

This section tests whether you read the solutions that we hand out.

**8**. [**10 points**] After pouring resources in extraterrestrial technologies, scientists have found a way to build a cloning machine in Area 51. Unfortunately, it turns out that alien technology is imperfect, so the scientists want to perform experiments with the machine.

Suppose that the cloning machine will successfully create one clone from any rabbit who is placed into it with probability $p$. With probability $1 - p$, the cloning machine fails and the rabbit who was placed into the machine is electrocuted and dies. As part of the experiment, the scientists begin with a single sample rabbit in room $A$. Each afternoon, they take all of rabbits who were originally in room $A$ and place each of them one by one into the cloning machine. For each rabbit who is placed into the machine, if the experiment is successful, the scientists take the original rabbit and its clone and place them into room $B$. If the experiment is unsuccessful, the scientists secretely dispose of the poor rabbit's mortal remains. Each evening, the scientists move all of the rabbits from room $B$ back to $A$. This experiment goes on until all of the rabbits die. What is the probability that this experiment ends?

**Basic Techniques.**

This part will test your ability to apply techniques that we have explicitly identified in lecture. You need to have practiced each technique enough to be able to handle small variations in the problems.

**9**. [**15 points**] A *biased* coin, which comes up heads with probability $p$, is tossed until we see either a **total** of 2 heads or a **total** of 2 tails. Answer the following questions.

- (5 pts) What is the sample space? (List all possible outcomes)

- (5 pts) In this experiment, what is the probability that we get consecutive heads?

- (5 pts) Now, *given that* the process went on for 3 flips, what is the (conditional) probability of getting consecutive heads?

**10**. **[15 points]** Recall that $\phi(n)$ is the Euler phi function.

- (10 pts) Given three distinct primes $p, q, r$, give an expression for $\phi(p \cdot q \cdot r)$.

- (10 pts) Give and prove an expression for $\phi(p^2)$, where $p$ is prime.

## A Moment's Thought!

This section tests your ability to think a bit more insightfully. You must give complete explanations of your answers.

**11**. **[20 points]** Professor Gupta and Professor Lafferty each independently and uniformly at random pick a subset of $n$ problems from a set of $n^2$ problems to put on this test. What is the probability a given problem is picked by *both professors*? What is the expected number of problems picked by *both professors*? (Please give clear reasons.)