# 15-251: Great Theoretical Ideas In Computer Science

## Homework 9 Warmup Solutions

a. *If a planar connected graph has* $7$ *vertices and* $10$ *edges, how many regions (faces) does it have?*

Using Euler's Formula $v - e + f = 2$ with $v = 7$ and $e = 10$, we have

$$7 - 10 + f = 2 \implies f = 5$$

Therefore, there are $5$ regions (faces) in this planar connected graph.

b. *Using induction, prove that every tree of* $n \geq 1$ *vertices must have* $n - 1$ *edges. (Recall the definition of a tree as a connected graph with no cycles.)*

**Base Case:** $n = 1$: there is only one tree with one vertex, and that has zero edges.

**Inductive Hypothesis:** Suppose that for some $k \geq 1$, all trees with $k$ vertices have $k - 1$ edges. Hence we assume statement

$$S_k = \text{“all trees with } k \text{ vertices have } k-1 \text{ edges.”}$$

**Inductive Step:** We want to prove statement

$$S_{k+1} = \text{“all trees with } k+1 \text{ vertices have } k \text{ edges.”}$$

To do this, take an arbitrary tree $T$ with $k + 1$ vertices. There are two cases:

- Case I: Suppose this tree has a vertex $v$ with degree 1—i.e., $T$ has only one edge incident to $v$. Remove vertex $v$ and the incident edge to get a tree $T'$ on $k$ vertices—note that the removal of the vertex and the edge maintains that the new tree $T'$ is connected, and it does not create new cycles. Hence we can apply the I.H. to $T'$ to infer that $T'$ has $k - 1$ edges. Now add the removed vertex back in using the same edge. This gives us back the tree $T$, and we have one more edge than in $T'$—hence tree $T$ has $k$ edges.
- Case II: Suppose all vertices have degree at least $2$ in $T$: we claim this is impossible if $T$ is a tree. Indeed, in this case we show $T$ must have a cycle, which is impossible. To prove this, set all edges to be initially unmarked. Pick any vertex $v_0$ and an edge incident to it, and go to a neighbor $v_1$, marking the edge $(v_0, v_1)$. Now choose any unmarked edge incident to $v_1$—there must be such an edge, since there are at least two edges incident to $v_1$—and traverse this to go to a neighbor $v_2$, again marking edge $(v_1, v_2)$. Keep repeating this: note that each time you reach a new vertex $v_i$, only the edge used to reach $v_i$ is marked, and hence there is an unmarked edge to leave the vertex. Since there are finitely many vertices, you will eventually see a vertex $v_j$ such that $v_j = v_i$ for some $i < j$. But now $v_i, v_{i+1}, \cdots, v_{j-1}, v_j$ is a cycle in $T$, which is a contradiction.

This shows that $T$, which is an arbitrary tree on $k + 1$ nodes, has $k$ edges. In turn, this proves statement $S_{k+1}$, and completes the induction.

**Note:** A popular strategy is to take a tree $T'$ with $k$ nodes and construct a tree $T$ with $k + 1$ nodes by adding an edge, and use the I.H. etc. Why is that a bad idea? Think about this before attempting the actual HW.