15-251: Great Theoretical Ideas In Computer Science

Homework 7 (due Thursday, October 15)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be in closed form.*

0. Warmup (0 points)

1. Consider the number $250!$. How many zeroes are there at the end of this number?
2. What is the smallest natural number n such that $n \equiv k - 1 \pmod{k}$ simultaneously for all of $k = 2, 3, 4, 5, 6, 7, 8, 9$, and 10?
3. Suppose n is the sum of two perfect squares. Show that $2n$ is also the sum of two perfect squares. (Recall a perfect square is a number a^2 for some integer a.)

1. Unfortunate Clones (15 points)

After pouring resources in extraterrestrial technologies, scientists have found a way to build a cloning machine in Area 51. Unfortunately, it turns out that alien technology is imperfect, so the scientists want to perform experiments with the machine.

Suppose that the cloning machine will successfully create one clone from any rabbit who is placed into it with probability p. With probability $1 - p$, the cloning machine fails and the rabbit who was placed into the machine is electrocuted and dies. As part of the experiment, the scientists begin with a single sample rabbit in room A. Each afternoon, they take all of rabbits who were originally in room A and place each of them one by one into the cloning machine. For each rabbit who is placed into the machine, if the experiment is successful, the scientists take the original rabbit and its clone and place them into room B. If the experiment is unsuccessful, the scientists secretely dispose of the poor rabbit's mortal remains. Each evening, the scientists move all of the rabbits from room B back to A. This experiment goes on until all of the rabbits die. What is the probability that this experiment ends?

2. Chess Tournament (20 points)

The Computer Science department decides to force three professors, Anupam Gupta, Ryan O'Donnell, and Luis Von Ahn to spend a summer in a 17-million dollar house in the Northeast, with one major caveat: the professors need to pay the cost of the house! Naturally, as academics, the three decide on a complex and convoluted way of determining who pays the cost: a tournament consisting of a series of chess games (with their bots, of course!). The rules are as follows:

1. Anupam picks the first two contestants.
2. The winner of any game plays the next one with the person who was left out.
3. The first person to win any two games wins the tournament.

Assume that Luis is the strongest contestant, Anupam is the weakest one, and the probability that each player wins against another player remains unchanged over the course of the tournament. Prove that Anupam has the highest chances of winning if he plays the first game with Ryan.

3. Combining Pairings (15 points)

Suppose S and T are two stable pairings of n boys with n girls. Suppose we want to create a new stable pairing M by combining S and T as follows: for each boy b, let g_s, g_t be the girls he is paired with in S and T, respectively. In M, pair him with his favorite of the two.

(a) *(5 points)* Prove that M is a valid pairing–that is, each girl is assigned to only one boy.

(b) *(5 points)* Prove that M is stable.

(c) *(5 points)* Use the definition of "optimal mates" and the above two parts to prove that there exists a male-optimal pairing: i.e., a stable pairing M^* where each boy is simultenaously paired with his optimal mate. (You may not use the fact that the TMA produces a male-optimal pairing.)

4. Number Theory Sampler (35 points)

(a) *(5 points)* Show that it is possible to generate n consecutive composite (non-prime) integers for any positive integer n.

(b) *(5 points)* Prove that an odd number $n > 1$ is prime if and only if it is not expressible as a sum of 3 or more consecutive positive integers

(c) *(5 points)* Let n be an integer greater than 2. Show that an even number of fractions

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{n-2}{n}, \frac{n-1}{n}$$

is irreducible.

(d) *(10 points)* Show that for any positive integers a and b, the number $(2a + b)(a + 2b)$ cannot be a power of 2.

(e) *(10 points)* Let a, b be two positive integers such that $GCD(a, b) = 1$. Show that $ab - a - b$ is the largest value that cannot be written in the form $ax + by$, where both x and y are non-negative integers.

5. Euclid and Fibonacci, Sitting in a Tree (15 points)

Recall the Fibonacci numbers are $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Suppose that a and b are positive integers such that $a > b > 0$.

(a) *(5 points)* For $n \geq 1$, how many divisions does Euclid's algorithm take to compute $GCD(F_{n+1}, F_n)$? You need to prove your answer.

(b) *(10 points)* Suppose $g(n)$ is the answer for (5a). Show that if $A > B$ are positive integers and Euclid's algorithm takes $g(n)$ divisions to compute $GCD(A, B)$, then $A \geq F_{n+1}$ and $B \geq F_n$.