15-251: Great Theoretical Ideas In Computer Science

Homework 7 Warmup Solutions

a. *Consider the number* $250!$ *How many zeroes are there at the end of this number?*

The number of zeroes at the end of $250!$ is the number of times $250!$ is divisible by 10. Because $10 = 2 \times 5$, the number of times $250!$ is divisible by 10 is equal to either the number of times 2 is a factor of $250!$, or the number of times 5 is a factor, whichever is smaller.

Count the number of times 5 is a factor of $250!$ Every multiple of 5 that is at most 250 contributes a factor 5 to $250!$ There are 50 such multiples of 5, so there are 50 factors. However, those multiples of 5 which are also multiples of $25 = 5^2$ each contribute an additional factor 5, for a total of 10 more factors. Finally, those multiples of 5^2 which are also multiples of $125 = 5^3$ each contribute a third factor 5, for a total of 2 more factors. Therefore, the number of times 5 is a factor of $250!$ is $50 + 10 + 2 = 62$.

There are 125 even numbers between 1 and 250, so 2 is a factor of $250!$ at least 125 times. Since there are only 62 factors 5 in $250!$, $250!$ is divisible by 10 only 62 times, and there are 62 trailing zeroes at the end of the number.

b. *What is the smallest natural number* n *such that* $n \equiv_k k - 1$ *simultaneously for each of* $k = 2, 3, 4, 5, 6, 7, 8, 9,$ *and* 10*?*

For each k, suppose $n \equiv_k k - 1$. Then, $n + 1 \equiv_k k \equiv_k 0$, so $k \mid n + 1$. We need the smallest number $n + 1$ that is divisible by all the possible values of k.

Consider the prime factorization of $n+1$. The exponent of 2 in $n+1$ must be at least 3 in order for $n + 1$ to be divisible by 8. The exponent of 3 must be at least 2 in order for $n + 1$ to be divisible by 9. The exponents of 5 and 7 must both be at least 1 in order for $n+1$ to be divisible by those numbers. Take the smallest number under these restrictions, i.e. $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. Note that this number is divisible by all the values of k, so $n + 1 = 2520$. Then, $n = 2519$.

c. *Suppose* n *is the sum of two perfect squares. Show that* $2n$ *is also the sum of two perfect squares. Recall a perfect square is a number* a^2 *for some integer* a.

Suppose n is the sum of two perfect squares, i.e. $n = a^2 + b^2$ for some integers a, b. Then,

$$\begin{aligned} 2n &= 2(a^2 + b^2) \\ &= (a^2 + b^2) + (a^2 + b^2) + 2ab - 2ab \\ &= (a^2 + 2ab + b^2) + (a^2 - 2ab + b^2) \\ &= (a + b)^2 + (a - b)^2 \end{aligned}$$

Therefore, since $a + b$ and $a - b$ are also integers, $2n$ is also the sum of two perfect squares.