15-251: Great Theoretical Ideas In Computer Science

Homework 6 (due Thursday, Oct 9)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

0. Warmup (0 points)

a. Suppose A and B are disjoint events. Are A and B independent?

b. If A is independent of B, is B definitely independent of A?

c. Under what condition(s) is $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$ true?

d. Under what condition(s) is $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ true?

e. Suppose $\Pr(A) = {}^1\!/_2$, $\Pr(B) = {}^1\!/_3$, and $\Pr(A \mid B) = {}^1\!/_4$. What is $\Pr(B \mid A)$?

f. What is $\Pr(A \cup \overline{A})$? $\Pr(A \cap \overline{A})$? What about $\Pr(A \mid B) \cdot \Pr(B) + \Pr(A \mid \overline{B}) \cdot \Pr(\overline{B})$?

g. Let X and Y be two random variables. Which of the following expressions make sense, and which are meaningless?

$\Pr(X)$	$\Pr(X = 3)$	$\Pr(X = 3 \mid Y)$
$\mathbb{E}[X]$	$\mathbb{E}[X = 3]$	$\mathbb{E}[X \mid Y]$
$\mathbb{E}[X \mid Y = 5]$	$\mathbb{E}[\mathbb{E}[Y]]$	$\mathbb{E}[\Pr(X = 3 \mid Y = 5)]$

h. If you roll k dice, what is the expected value of their sum?

1. Expect the Unexpected (10 points)

Let X be a random variable that can take on non-negative, integer values, such that the expectation $\mathbb{E}[X]$ is finite. Show that

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr(X \geq i)$$

2. My Head is Spinning (25 points)

A dial has n regions labeled from 1 to n, in clockwise order, with a pointer that, when spun, is equally likely to land on any of these regions. When the pointer lands on region k, you win 2^k dollars.

a. *(2 points)* What is the amount of money you should expect to win from one spin?

b. *(5 points)* Suppose that you also have the following option: after your spin, you may choose to either leave the pointer where it is, or flip an unbiased coin and move the pointer one spot clockwise if the coin shows "heads", or one spot counterclockwise if the coin shows "tails". Again, you win 2^k dollars if the final position of the pointer is on region k. Assuming you are trying to maximize your expected winnings, for which values of k should you take the coin flip? What is the expected outcome under this strategy?

c. *(5 points)* Suppose you now have the following option instead of the one above: If you don't like the value of the first spin, you can spin the pointer once more, and take the outcome from the second spin (but you can no longer flip the coin). For which values of k should you take the second spin? What is the expected outcome under this strategy?

d. *(13 points)* Suppose now you can take neither, one, or both of the above options, in any order. Specifically, you can:

   (a) Take the value of the first spin;
   (b) Flip the coin on the first spin and take the outcome of the flip;
   (c) Spin again without flipping the coin, and take the value of the second spin;
   (d) Flip the coin, then spin again if you don't like the outcome from the coin;
   (e) or spin again, then flip the coin to adjust the outcome if you don't like the outcome from the second spin.

   Now what is the optimal strategy, and what is its expected outcome?

3. Test-Taking Strategy (20 points)

The course staff is considering a different procedure for administering tests: you can answer the questions in any order, but as soon as you answer a question incorrectly, the test terminates and your grade is the sum of the scores you have earned up to that point. We would also cease to award partial credit.

To prepare yourselves for the event that we do decide to use this procedure, your job is to determine the order in which you should answer the questions to maximize your expected score. Say there are n questions, question i has point value v_i, and the probability that you answer it correctly is p_i. Show that the ordering of the questions that maximizes your expected score is to answer them in nonincreasing order of $\frac{v_i p_i}{1-p_i}$.

Note: we are not likely to actually switch to this procedure.

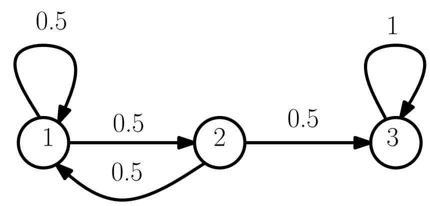
4. Markov Chains (15 points)

A *Markov Chain* is a set of states labeled $1, 2, \ldots, n$, together with a transition probability p_{ij} for each pair of vertices i, j that satisfy

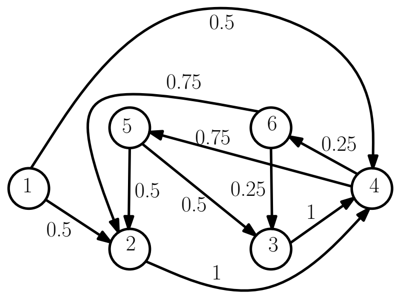
- $0 \leq p_{ij} \leq 1$ for each pair i, j of states, and
- $\sum_{j=1}^{n} p_{ij} = 1$ for every i.

Note that p_{ij} and p_{ji} may be unequal. Consider a walk on a Markov Chain as follows: at time $t = 0$, you are in state 1. If you are at state i at time t, at time $t+1$ you move to state j with probability $p_{i,j}$.

a. *(6 points)* In the Markov Chain below, what is the expected number of steps taken before reaching state 3? The numbers on the arrows are the transition probabilities; e.g., the probability $p_{23} = 0.5$ and $p_{32} = 0$ (since there is no arrow from 3 to 2 in the diagram).



b. *(9 points)* In the Markov Chain below, what is the probability that you are in state 2 after n steps? What is the expected number of times you are in state 2 out of the first n steps?



5. Mediocre Magician (30 points)

As some of you may have guessed by now, Dan is a powerful magician. However, due to the stresses of being a TA, he hasn't had time to practice his magic, and is a little rusty with his spells. This problem will analyze the properties of some of his magical talents.

a. *(10 points)* Dan has k hats, labelled 1 through k. In order to pull rabbits out of these hats, the rabbits must first go into the hats. He can conjure any number of rabbits, but as soon as a rabbit is conjured, it chooses a hat at random and jumps into it. What is the expected number of rabbits that Dan must conjure before there is a rabbit in each hat?

b. *(10 points)* Dan frequently flies around the world at night. Every minute, he flies $\frac{1}{n}$ of the way around the world, staying over the equator. (I.e. if he flew in a straight line, it would take n minutes to circumnavigate the world). However, Dan cannot control which direction he goes, and every minute, he randomly moves East or West. What is the expected amount of time it takes Dan to reach the other side of the world? You may assume n is even.

c. *(10 points)* After failing to properly control his rabbits, two of them escape—one red, one blue. Since these are magical rabbits, they don't reproduce according to the Fibonacci numbers. Instead, they behave as follows:

  (a) At time $t = 1$, the red rabbit escapes.
  (b) At time $t = 2$, the blue rabbit escapes.
  (c) For $t > 2$, exactly one of the free rabbits at time $(t - 1)$ clones itself, with this rabbit being chosen uniformly at random.

For $k \in \{1, 2, \ldots, t-1\}$, and time $t > 1$, let Q_{kt} be the probability that there are k free red rabbits at time t. Show that $Q_{kt} = \frac{1}{t-1}$.