15-251: Great Theoretical Ideas In Computer Science

Homework 6 Warmup Solutions

0. Warmup (0 points)

a. *Suppose A and B are disjoint events. Are A and B independent?*

Only if $\Pr(A) = \Pr(B) = 0$. Otherwise, $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(\emptyset)}{\Pr(B)} = 0 \neq \Pr(A)$. Similarly for $\Pr(() B \mid A)$.

b. *If A is independent of B, is B definitely independent of A?*

Yes; If $\Pr(A) = \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$, then $\Pr(B) = \frac{\Pr(A \cap B)}{\Pr(A)} = \Pr(B \mid A)$.

c. *Under what condition(s) is $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$ true?*

This is the definition of independence.

d. *Under what condition(s) is $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ true?*

When A and B are disjoint.

e. *Suppose $\Pr(A) = 1/2$, $\Pr(B) = 1/3$, and $\Pr(A \mid B) = 1/4$. What is $\Pr(B \mid A)$?*

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{\Pr(B \cap A)}{\Pr(A)} \cdot \frac{\Pr(B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(A)} \cdot \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(A)} \cdot \Pr(A \mid B) = \frac{1}{6}$$

f. *What is $\Pr\left(A \cup \overline{A}\right)$? $\Pr\left(A \cap \overline{A}\right)$? What about $\Pr(A \mid B) \cdot \Pr(B) + \Pr\left(A \mid \overline{B}\right) \cdot \Pr\left(\overline{B}\right)$?*

$\Pr\left(A \cup \overline{A}\right) = 1?$; $\Pr\left(A \cap \overline{A}\right) = 0$; $\Pr(A \mid B) \cdot \Pr(B) + \Pr\left(A \mid \overline{B}\right) \cdot \Pr\left(\overline{B}\right) = \Pr(A)$ - this just partitions the ways for A to occur based on whether B occured or not.

g. Let X and Y be two random variables. Which of the following expressions make sense, and which are meaningless?

$\Pr(X)$	$\Pr(X = 3)$	$\Pr(X = 3 \mid Y)$
$\mathbb{E}[X]$	$\mathbb{E}[X = 3]$	$\mathbb{E}[X \mid Y]$
$\mathbb{E}[X \mid Y = 5]$	$\mathbb{E}[\mathbb{E}[Y]]$	$\mathbb{E}[\Pr(X = 3 \mid Y = 5)]$

It does not make sense to talk about the "probabillity" of a random variable. It does not make sense to talk about the expected value of an event. It does not make sense to talk about the expected value of a random variable "given" another random variable, but it does make sense to talk about the expected value of a random variable given an *event* involving another random variable. "$Y = 3$" is an event. If Y is a random variable, then $\mathbb{E}[Y]$ is another random variable. If X and Y are random variables, then $\Pr(X = 3 \mid Y = 5)$ is another random variable. Thus, the ones that make sense are $\Pr(X = 3)$, $\mathbb{E}[X]$, $\mathbb{E}[X \mid Y = 5]$, $\mathbb{E}[\mathbb{E}[Y]]$, and $\mathbb{E}[\Pr(X = 3 \mid Y = 5)]$.

h. *If you roll k dice, what is the expected value of their sum?*

The expected value of *one* die is 3.5. By linearity of expectation, then, the expected value of the sum is $k \cdot 3.5$.