15-251: GTI Test 1 Practice (Solutions)

Name:

AndrewID:

Section:

INSTRUCTIONS. For the first six problems you do not need to give any reason or justification for your answer. For the other problems you must give a complete explanation. WRITE CLEARLY. WRITE YOUR NAME, ANDREW ID AND SECTION IN THE BOX ABOVE. This is closed book quiz. You may not use notes. You may use a calculator, if you wish.

There is a total of 9 pages on this test, make sure none are missing.

Problem	Points	Score	Problem	Points	Score
1	5		8	10	
2	5		9	15	
3	5		10	15	
4	5		11	25	
5	5		Σ	100	
6	5				
7	5				

Repeat After Me.

This part is to test your ability to regurgitate basic facts. You should either have these facts memorized or be able to re-derive them on the spot.

1. [5 points]
What is $\sum_{k=0}^{n} X^k$ in closed form?

$$\frac{X^{n+1} - 1}{X - 1}$$

2. [5 points]
How many ways are there to distribute n *indistinguishable* pieces of gold amongst k *distinguishable* pirates?

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

3. [5 points]
How many digits does n have in base b?

$$\lfloor \log_b n \rfloor + 1$$

4. [5 points]
How many ways are there to rearrange the letters in the word HAKUNAMATATA?

$$\frac{12!}{5! \cdot 2!}$$

5. [5 points]
Let A be a set of n elements. How many subsets of A have odd size? Briefly explain your answer.

2^{n-1}. There are 2^n subsets of A, and the odd-sized subsets of A can be put into a one-to-one correspondence with the even-sized subsets. Therefore, half are odd and half are even.

6. [5 points]
If $f : A \to B$ is a 1-1 function, then what can you correctly infer: the size of A *(a) smaller than or equal to, (b) equal to, or (c) larger than or equal to* the size of B.

$$|A| \leq |B|$$

7. [5 points]
What's wrong with the following argument for counting the number of poker hands of 5 cards with *at least* 3 cards of a single rank (eg, {3 Aces, a Queen, and a 5} or {4 Kings and an Ace})?

- Choose a rank for the 3-of-a-kind: $\binom{13}{1}$
- Choose three of four suits to use in the 3-of-a-kind: $\binom{4}{3}$
- Choose the fourth and fifth cards from the remaining ones: $\binom{49}{2}$
- Thus, the final answer is the product of the above: $\binom{13}{1}\binom{4}{3}\binom{49}{2}$

The hand $A\heartsuit$ $A\diamondsuit$ $A\spadesuit$ $A\clubsuit$ $K\spadesuit$ (among others), is counted more than once. Here are two ways to count this hand: both start by choosing the ace rank for the 3-of-a-kind. In the first way, the suits chosen for the 3-of-a-kind are $\heartsuit$ $\diamondsuit$ $\spadesuit$. Then, the ace of clubs ($A\clubsuit$) is chosen as the fourth or fifth card. In the other way, the suits chosen are $\diamondsuit$ $\spadesuit$ $\clubsuit$, and $A\heartsuit$ is chosen as one of the last two cards.

Reading Solutions.

This section tests whether you read the solutions we hand out.

8. [10 points]

Consider all $(2n)!$ orderings of this set, with lines drawn every two ele counts every partition, but we have overcounted by 2^n times, for the rearra there are $\boxed{(2n)!/2^n}$ possible divisions.

Basic Techniques.

This part will test your ability to apply techniques explicitly identified in lecture. You need to have practiced each technique enough to be able to handle small variations in the problems.

9. [15 points]
Find and prove a closed-form formula for the sequence a_n, defined by $a_0 = 1$ and $a_n = 3a_{n-1}$.

Claim: $a_n = 3^n$ for all $n \geq 0$. Proof by induction on n:

- *Base case:* $n = 0$: $a_0 = 1 = 3^0$.
- *Inductive hypothesis:* Suppose $a_n = 3^n$ for *some* $n \geq 0$.
- *Inductive step:* Then, consider a_{n+1}:

$$\begin{aligned} a_{n+1} &= a_{n+1} = 3 \cdot a_n && \text{[definition of } a_n\text{]} \\ &= 3 \cdot 3^n && \text{[inductive hypothesis]} \\ &= 3^{n+1} && \text{[algebra]} \end{aligned}$$

Therefore, the statement holds by induction for all $n \geq 0$.

10. [15 points]
Let F_n be the n^{th} Fibonacci number. Using induction, prove that for any integer k, F_n divides F_{kn}.
Hint: recall that $F_{a+b} = F_a \cdot F_{b+1} + F_{a-1} \cdot F_b$.

The proof will be by induction on k.

- *Base case:* $k = 0$: $F_0 = 0$, and any number divides zero. So, $F_n | F_{0 \cdot n}$.
- *Inductive hypothesis:* Suppose the claim is true for *some* $k \geq 0$: $F_n | F_{kn}$. That is, there exists an integer d such that $d \cdot F_n = F_{kn}$.
- *Inductive step:* Consider $F_{(k+1)n}$:

$$\begin{aligned} F_{(k+1)n} &= F_{kn+n} && \text{[algebra]} \\ &= F_{kn} \cdot F_{n+1} + F_{kn-1} \cdot F_n && \text{[using the hint]} \\ &= d \cdot F_n \cdot F_{n+1} + F_{kn-1} \cdot F_n && \text{[inductive hypothesis]} \\ &= F_n(d \cdot F_{n+1} + F_{kn-1}) && \text{[algebra]} \end{aligned}$$

Therefore, F_n divides $F_{(k+1)n}$, and the statement is true of all k by induction.

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers.

11. [25 points]
Consider the following **variant** on the Towers of Hanoi. We have three pegs named A, B, and C, and n delicate disks stacked on peg A peg that we want to move to peg C. You cannot put a bigger disk on top of a smaller disk; *you can only move disks from peg A to peg B, or peg B to peg C, or peg C to peg A - you can't move disks from peg A to peg C*. Show *upper* and *lower* bounds for $f(n)$, the number of moves it takes to transfer n discs from peg A to peg C.

1. To show a lower bound, prove that *any* algorithm takes at least that many moves. Show the easy lower bound $f(n) \geq n$.

   Only one disk can be moved at each iteration, and there are n disks to move. Therefore, $f(n) \geq n$.

2. To show an upper bound, give *an* algorithm and analyze the runtime. Here, show a recursive algorithm and give a recurrence for the runtime; there is no need to find a closed form for the runtime.

   (a) Base Case: What do you do if there is exactly 1 disk on peg A? How many moves does it take?

   It takes two moves, from A to B and from B to C, to get one disk from A to C. Therefore, $f(1) = 2$.

(b) Recursive Case: Assume you know how to move n disks. What do you do if you have $n+1$ disks? What is $f(n+1)$ in terms of $f(n)$?

Here is one way to do this:

- Recursively move the top n disks on peg A to peg C. By assumption, this takes $f(n)$ moves.
- Move the largest (bottom) disk on peg A to beg B. This is one move.
- Move all the n disks on peg C back to peg A. This can be done by recursively moving all the n disks to peg B (through peg A), and then from there to peg A (through peg C). This takes $2f(n)$ moves.
- Move the largest disk, which is now on peg B, to peg C. This is one move.
- Move the n disks on peg A back to peg C and on top of the largest disk — $f(n)$ more moves.

Therefore, the recursive case of the recurrence, for the algorithm given above, is

$$f(n+1) = 4f(n) + 2$$