15-251: GTI Practice Test Fall 2009

Name:

Andrew ID:

Section:

INSTRUCTIONS:

- WRITE YOUR NAME, ANDREW ID AND SECTION IN THE BOX ABOVE.
- This is a closed book test. You may *not* use notes. You may *not* use a calculator.
- For problems 1–8, you do not need to give any reason or justification for your answers. However, partial credit may not be given unless you show your work.
- For the other problems you must give clear and complete proofs. In your proofs, you may quote and use any results presented during lectures; all other steps and results REQUIRE PROOFS. Please write clearly.
- **Unless specified otherwise, please give closed forms, and simplify expressions as much as possible.**

There is a total of 11 pages on this test; make sure none are missing.

Problem	Points	Score	Problem	Points	Score
1	5		8	5	
2	5		9	10	
3	5		10	15	
4	5		11	15	
5	5		12	20	
6	5		Σ	100	
7	5				

Repeat After Me.

This part is to test your ability to regurgitate basic facts. You should either have these facts memorized or be able to re-derive them on the spot.

1. [5 points]
Give a closed form expression for F_n, the n^{th} Fibonacci number. Recall that $F_0 = 0, F_1 = 1$.

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2. [5 points]
How many ways are there of arranging 10 pancakes into 3 or fewer stacks? (The pancakes are identical, and hence the order of the pancakes in each stack does not matter.)

3. [5 points]
How many ways are there to rearrange the 7-character string "FOO BAR" so that the space stays in the middle? For example, one such rearrangement is "BOO FAR".

4. [5 points]
Give a stack of 10 pancakes that requires at least 10 flips to sort.

5. [5 points]
Find a closed form for the sum

$$-m + (-m+1) + (-m+2) + \cdots + (-1) + 0 + 1 + 2 + \cdots n$$

where $n \geq 0$ and $m \geq 0$ are positive integers.

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6. [5 points]
An *n-trit ternary* representation of a number x is $t_{n-1}t_{n-2}\ldots t_1t_0$, where

$$x = t_0 + t_1 3 + t_2 3^2 + t_3 3^3 + \cdots t_{n-1} 3^{n-1}.$$

Each $t_i \in \{0, 1, 2\}$ is called a *trit*. What is the largest number you can represent in ternary using n trits?

7. [5 points]
How many ways are there for 7 students to take one book each from a collection of 15 different books? (Each student takes and keeps **exactly** one book.)

8. [5 points]
Find the nim-sum of 34, 42, and 50.

Reading Solutions.

This section tests whether you read the solutions we hand out. It is an opportunity to redeem yourself for past misdeeds.

9. [10 points]
Define $a_1 := 1$ and $a_2 := 2$. Additionally, define $\forall\ n \geq 3$:

$$a_n := 2a_{n-1} - a_{n-2}$$

Prove that $\forall\ n \geq 1$:

$$a_n \geq n - 1$$

Basic Techniques.

This part will test your ability to apply techniques explicitly identified in lecture. You need to have practiced each technique enough to be able to handle small variations in the problems.

10. [15 points]
Consider the following definition for the function $f(x)$, defined only for powers of 2:

$$f(x) = 2f(x/2) + 1$$
$$f(1) = 1.$$

(a) Calculate $f(1)$, $f(2)$, $f(4)$ and $f(8)$.

$f(1) =$ ______ $f(2) =$ ______ $f(4) =$ ______ $f(8) =$ ______

(b) Find a closed form for the function $f(x)$.

$f(x) =$ ____________________

(c) Prove your answer correct.

11. [15 points]
(Fun with Squares and Triangles.) Prove that

$$\sum_{i=1}^{2n} \triangle_i = \sum_{i=1}^{n} \square_{2i}.$$

That is,

$$\sum_{i=1}^{2n} \frac{i(i+1)}{2} = \sum_{i=1}^{n} (2i)^2 \,.$$

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers.

12. [20 points]
(Nested Sets.)

(a) (13 points) How many ways are there of constructing $n+1$ **distinct** sets such that

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$$

and each $A_i \subseteq \{1, 2, \ldots, n\}$. Recall that $X \subseteq Y$ just means that X is a subset of Y.

For example, if $n = 4$, one possible way to construct this sequence of 5 sets is $A_0 = \emptyset$, $A_1 = \{2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 4\}$, and $A_4 = \{1, 2, 3, 4\}$.

(part **(b)** on next page)

(b) (7 points) How many ways are there of constructing $n + 1$ **not necessarily distinct** nested subsets

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$$

and each $A_i \subseteq \{1, 2, \ldots, n\}$.

For example, if $n = 4$, a possible sequence is $A_0 = \{2\}$, $A_1 = \{2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 4\}$, and $A_4 = \{1, 2, 4\}$.