

15-251: Great Theoretical Ideas In Computer Science

Homework 5 (due Thursday, October 1)

Directions: Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.

0. Warmup (0 points)

- a. Given that $A(x)$ is the generating function for $\langle a_0, a_1, a_2, a_3, \dots \rangle$, find generating functions for:
- (a) $\langle a_0 + a_1, a_1 + a_2, a_2 + a_3, \dots \rangle$
 - (b) $\langle a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \dots \rangle$
 - (c) $\langle a_0, a_1b, a_2b^2, a_3b^3, \dots \rangle$ Where b is a constant
 - (d) $\langle a_0, 0, a_2, 0, a_4, 0, \dots \rangle$
- b. Given any $n + 1$ distinct integers between 1 and $2n$, show that one of them is divisible by another.

1. Practice Test (40 points)

The first part of this assignment will help you prepare for Test 1, which will be administered in recitation on Monday, September 28th. A copy of a practice test can be obtained via the link on the course calendar: your score on this problem will be your score on the practice test multiplied by $\frac{2}{5}$.

Treat this practice test like you would any homework problem. However, you should keep in mind that you only have 50 minutes to complete the real test when you take it in recitation. Your solutions on the practice test may be handwritten — they do NOT have to be typeset. **The completed practice test is due at the beginning of recitation on Monday, September 28th.**

2. Use the Pigeonhole Principle (25 points)

- a. (10 points) You are given 4×7 grid with each cell colored black or white, but this coloring is arbitrary (and not necessarily in a checkerboard pattern). Show that it is possible to select a non-trivial rectangle (having length and width at least 2 each) such that the four corner cells of this rectangle are of the same color.
- b. (15 points) A poll of 21 students at the University of PHP finds that between the 21 students, there are a total of 15 relationships, with each student participating in at least one relationship. (A relationship is between two people.) Prove that within these 21 students, there must be a set of 6 relationships that involve 12 distinct students.

3. Continued Fractions (10 points)

Prove that for any integer $x > 0$,

$$x + \frac{1}{2x + \frac{1}{2x + \frac{1}{2x + \dots}}}$$

is a continued fraction representation of $\sqrt{x^2 + 1}$.

4. Terrible Graders (25 points)

Since Brad is deluged with 251 solutions, he has been contemplating assigning random *integer* scores between 0 and 100 to the students in the class; there are n students. To ensure that the class average is consistent with what the professors want, he wants these n scores sum to some given integer $k > 0$.

- For full (25) points, find the number of ways Brad can distribute the points (as a function of n and k). You may not be able to get a closed-form solution; please simplify as much as possible.
- Or, for 15 points, give a solution for the case $n = 5$ and $k = 350$
- Or, for 8 points, solve for the case $n = 5$ and $k = 400$.

In all cases, give full proofs. Clearly, if you find a solution to the any part, you need not solve the following parts.