15-251: Great Theoretical Ideas In Computer Science

Homework 5 Warmup Solutions

0. Warmups (0 points)

a. Given that $A(x)$ is the generating function for $\langle a_0, a_1, a_2, a_3, \ldots \rangle$, find generating functions for:

(a) $\langle a_0 + a_1, a_1 + a_2, a_2 + a_3, \cdots \rangle$

Note that

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

and hence

$$B(x) = \frac{1}{x}(A(x) - a_0) = a_1 + a_2 x + a_3 x^2 + \ldots$$

and so

$$A(x) + B(x) = \frac{A(x)(1+x) - a_0}{x} = \sum_{i \geq 0} (a_i + a_{i+1}) x^i$$

is the answer.

(b) $\langle a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \cdots \rangle$

This is directly from Lecture 8, Slide 45: the answer is $\frac{A(x)}{1-x}$.

(c) $\langle a_0, a_1 b, a_2 b^2, a_3 b^3, \cdots \rangle$ Where b is a constant

We want $\sum_{i \geq 0} a_i b^i x^i$, but this is just $A(bx)$.

(d) $\langle a_0, 0, a_2, 0, a_4, 0, \cdots \rangle$

Setting $b = -1$ in the expression above, we get $A(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a^4 x^4 \ldots$. Hence

$$\frac{A(x) + A(-x)}{2} = \sum_{i \geq 0} \frac{1}{2}(a_i + (-1)^i a_i) x^i = a_0 + a_2 x^2 + \ldots.$$

b. Given any $n+1$ distinct integers between 1 and $2n$, show that one of them is divisible by another.

This is an application of the pigeonhole principle. Let A denote the set $\{1, 2, \ldots, 2n\}$. For each odd number $o \in A$, consider the set $A_o = A \cap \{o \cdot 2^i \mid i \in \mathbb{N}\}$. E.g., if $n = 4$, then $A_1 = \{1, 2, 4, 8\}$, $A_3 = \{3, 6\}$, $A_5 = \{5\}$ and $A_7 = \{7\}$. There are n such sets A_o, and we claim that these sets partition A: indeed, each number in A is a power of 2 times some odd number, and hence falls into exactly one of these parts.

Now we choose $n+1$ numbers, and there are n sets A_o, so by the pigeonhole principle, two of these numbers fall into the same set A_o for some odd number o. But then one of them divides the other.