15-251: Great Theoretical Ideas In Computer Science

Homework 4 (due Thursday, September 24)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

0. Warmup (0 points)

a. How many integer-valued solutions are there for the equation $x + y + z = 100$ where $x \geq 0, y \geq 0, z \geq 0$? What about the case when $x \geq 12, y \geq 19, z \geq -10$? What about the number of ways to distribute 100 bars of gold to 3 pirates *when not all bars of gold have to be given away*?

b. How many numbers between 1000 and 9999 have all distinct digits? How many of them are even?

c. How many ways are there of putting n *distinct* balls into m distinct bins? How many ways if the balls are *identical*?

d. How many different words can be created by rearranging the letters of $ISCREAMYOUSCREAM$?

e. What is $\binom{n}{0}2^n - \binom{n}{1}2^{n-1}3^1 + \binom{n}{2}2^{n-2}3^2 - \ldots + (-1)^n\binom{n}{n-1}2^1 3^{n-1} + (-1)^n\binom{n}{n}3^n$?

f. Given a set S of n items, how many ways are there to choose a pair of subsets (A, B) such that $A \subseteq B \subseteq S$? What if you want $A \subsetneq B \subsetneq S$?

g. Determine a generating function for the series: $\{0, F_2, 0, F_4, 0, F_6, 0, F_8, \cdots\}$, where F_i represents the i'th Fibonacci number.

1. Solving recurrences (20 points)

Consider the sequence defined by $f_0 = 2$, $f_1 = 9$, and $f_n = 6f_{n-1} - 9f_{n-2}$.

a. *(5 points)* Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$. By multiplying both sides of the recurrence by x^n and summing both sides, find a closed formula for $F(x)$.

   Hint: $F(x)$ should be of the form $\frac{\alpha x + \beta}{\gamma x^2 + \delta x + \varepsilon}$.

b. *(5 points)* Using partial fractions, split $F(x)$ into two terms, each of which has a linear function raised to an integer power as its denominator.

c. *(5 points)* Using generating functions, convert those two terms into infinite sums. Combine them into one sum.

d. *(5 points)* Noting that $F(x)$ is both equal to $\sum_{n=0}^{\infty} f_n x^n$ and your sum from part (c), find a closed-form formula for f_n.

2. Banana Split (or, Bob goes Bananas) (20 points)

Bob goes to the market and buys an even number each of apples and bananas, which he lays down in a circle. Alice claims that she can choose two positions on the circle such that both the clockwise and counterclockwise portions of the circle between those two positions have the exactly half the apples and half the bananas. Prove that Alice is correct, no matter how Bob lays down the fruit.

3. I See a Circle as Pretty as a Tree (20 points)

A *rooted binary tree* of n nodes is a root node connected to some left-subtree and some right-subtree such that the number of nodes in the left and right subtrees add up to $n-1$. Two trees are the same if their root nodes connect to the same left-subtree and same right-subtree. There are exactly 5 distinct rooted binary trees with 3 nodes.

```
     o           o       o       o       o
    / \         /       /         \       \
   o   o       o       o           o       o
              /         \         /         \
             o           o       o           o
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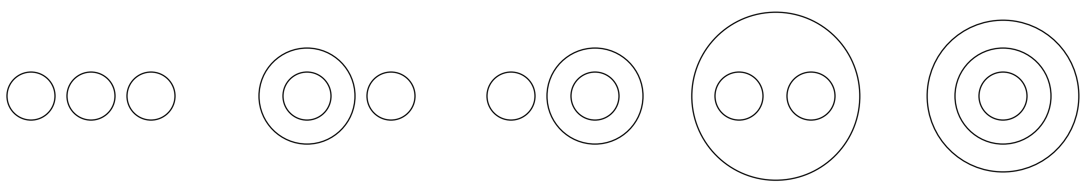
A computer scientist decides to make drawings resembling groups of circles. Each painting consists of a group of n circles such that (1) the centers of the circles are all collinear (it is possible to draw a line between the centers of all of the circles) (2) no two circles can intersect (3) the sizes of the circles are irrelevant. For example, there are a total of 5 drawings with 3 circles:



It turns out there is a correspondence between rooted binary trees and circle drawings. Your task is to show this correspondence (and prove it correct). To do this, for all $n \geq 1$:

a. *(10 points)* Give an algorithm that can be used to construct a unique circle drawing of n circles from any rooted binary tree of n nodes.

b. *(10 points)* Give an algorithm that can be used to construct a unique rooted binary tree of n nodes from any circle drawing of n circles.

4. Crazy Pascal (20 points)

Under the influence of too much chamomile tea, Pascal made another triangle where the outside numbers are 1, 2, 3, 4 .... Let j, k be positive integers such that $1 \leq j \leq k$. We define the value of the jth element in the kth row to be $CrazyPas(k, j)$, where:

$$CrazyPas(k, 1) = CrazyPas(k, k) = k$$

$$CrazyPas(k, j) = CrazyPas(k-1, j-1) + CrazyPas(k-1, j), \forall 1 < j < k$$

The first four lines of crazy Pascal's triangle are shown below. On the left, we start by writing the positive integers down the sides. On the right, we fill in the rest of the triangle using the same rules as Pascal's.

```
        1                      1
      2   2                  2   2
    3   _   3              3   4   3
  4   _   _   4          4   7   7   4
5   _   _   _   5      5  11  14  11    5
       ...                    ...
```

a) **(12 points)** Find and prove a closed-form formula for $CrazyPas(k, j)$ for all $1 \le j \le k$.

b) **(8 points)** Find and prove a closed-form for the row sums $R(k)$, where

$$R(k) = \sum_{i=1}^{k} CrazyPas(k, i).$$

5. Pyramids (20 points)

We will define an n-dimensional hyperpyramid of balls recursively as follows:

- A zero-dimensional hyperpyramid contains one ball, no matter how many levels it has.
- Each level of an n-dimensional hyperpyramid is an $n-1$ dimensional hyperpyramid. A hyperpyramid with m levels thus has m sub-hyperpyramids, which have $1, 2, \ldots, m$ levels apiece.

This definition is a bit confusing at first, so here are some examples:

(a) An m-level one-dimensional hyperpyramid. Each level is a zero-dimensional hyperpyramid, and thus is just a ball; hence, the m-level one-dimensional hyperpyramid is simply a line of m balls.

(b) An m-level two-dimensional hyperpyramid. Each level is a one-dimensional hyperpyramid, and thus is a line of balls; hence, the m-level one-dimensional hyperpyramid contains lines of 1 through m balls. Thus, it is a triangle of balls.

(c) A 3-level three-dimensional hyperpyramid. The first level is a two-dimensional three-level hyperpyramid; this is a triangle with rows 1,2 and 3. The second level is a two-dimensional two-level hyperpyramid; this is a triangle with rows 1 and 2. The third level is a two-dimensional one-level hyperpyramid; this is a single ball. Hence, this is just a triangular pyramid of height 3.

Find the following:

a. *(5 points)* The number of balls in an n-dimensional, 2-level hyperpyramid. Give a closed form.

b. *(5 points)* The number of balls in a 4-dimensional, m-level hyperpyramid. Give a closed form.

c. *(10 points)* The number of balls in an n-dimensional, m-level hyperpyramid. Give a closed form.