15-251: Great Theoretical Ideas In Computer Science

Homework 4 Warmup Solutions

0. Warmups (0 points)

a. *How many integer-valued solutions are there for the equation $x + y + z = 100$ where $x \geq 0, y \geq 0, z \geq 0$? What about the case when $x \geq 12, y \geq 19, z \geq -10$? What about the number of ways to distribute 100 bars of gold to 3 pirates when not all bars of gold have to be given away?*

The first part is simply a repackaging of pirates and gold. The answer is $\binom{102}{2}$, the number of ways of dividing 100 gold pieces between 3 pirates, or that of placing 2 dividers in a row of 102 items. By setting $x' = x - 12, y' = y - 19, z' = z + 10$, we get a correspondence between the valid solutions of $x + y + z = 100$ and all non-negative integer solutions of $x' + y' + z' = 79$, and the number of solutions to this new system is $\binom{79+3-1}{2}$. For the final part, this is like a problem where we consider an imaginary pirate, who takes away the gold not given to the other pirates—hence this is pirates and gold, now with 4 pirates and 100 pieces! The solution to this problem is therefore $\binom{103}{3}$.

b. *How many numbers between 1000 and 9999 have all distinct digits? How many of them are even?*

For the first part, the first digit can be any of $1, 2, \cdots, 9$ - giving us 9 choices for level 1 in the choice tree. The second digit can be any of the 10 digits, except for the one used up by the first digit, again giving it 9 choices. The third digit can be any of the 10 digits, excepts for the ones used up by the first and second digits, giving it 8 choices. The last digit can be any of the 10 digits, excepts for the ones used up by the first three digits, giving it 7 choices. Thus the required number is $9 \times 9 \times 8 \times 7 = 4536$.

For the second part, we need to work in a different order: The final digit has 5 choices, one of $0, 2, 4, 6, 8$. There are two cases. In the first case, the last digit was not a zero, and so the first digit now has 8 choices, any of $1, 2, \cdots 9$ except the one chosen for the final digit. The second digit also has 8 choices, any of $0, 1, 2, \cdots 9$ except the ones chosen for the first and last digit. The third digit has 7 choices, any of $0, 1, 2, \cdots 9$ except the ones chosen for the other 3 digits. Thus the required number in this case is $4 \times 8 \times 8 \times 7 = 2240$. The second case is when the last digit is zero, in which case the first digit has 9 choices, the second has 8 and third has 7. Summing over the two (disjoint) cases, we get the total number to be $4 \times 8 \times 8 \times 7 + 1 \times 9 \times 8 \times 7 = 2296$.

c. *How many ways are there of putting n* distinct *balls into m distinct bins? How many ways if the balls are* identical*?*

In the first part, there are m possible ways to choose for the location of each ball, and there are n choices to be made. The answer is thus m^n. If the balls are identical, it's yet another repackaging of the pirates and gold problem: n gold coins, m pirates: $\binom{n+m-1}{m-1}$ ways.

d. *How many different words can be created by rearranging the letters of $ISCREAMYOUSCREAM$?*

This is similar to the problem done in class. Let's think of all the letters being distinct: I_1 S_1 C_1 R_1 E_1 A_1 M_1 Y O U S_2 C_2 R_2 E_2 A_2 M_2. The number of arrangements of the new bunch of letters is $16!$. However, in each of these arrangements, rearranging the repeat letters amongst themselves gives rise to the same word! Therefore, we divide this answer by $2! \times 2! \times 2! \times 2! \times 2! \times 2!$—Thus the answer is $\frac{16!}{2^6}$.

e. *What is* $\binom{n}{0}2^n - \binom{n}{1}2^{n-1}3^1 + \binom{n}{2}2^{n-2}3^2 - \ldots + (-1)^{n-1}\binom{n}{n-1}2^1 3^{n-1} + (-1)^n\binom{n}{n}3^n$*?*

This is the term-wise expansion of $(2+(-3))^n$. The answer is therefore $(-1)^n$.

f. *Given a set* S *of* n *items, how many ways are there to choose a pair of subsets* (A, B) *such that* $A \subseteq B \subseteq S$*? What if you want* $A \subsetneq B \subsetneq S$*?*

Part 1: Let's choose a subset B of size k first. We know that from this subset of size k, there are 2^k ways of choosing a subset A of B. Thus, the total number of such $A \subseteq B \subseteq S$ is simply the sum $\sum_{k=0}^{n}\binom{n}{k}2^k$ which is the binomial series $(1+2)^n$ in disguise! The answer is 3^n.

Alternate solution: For each element, we have 3 choices: select it in sets A and B, select it in set B only, and select it in neither. The answer is thus 3^n.

Part 2: we want to choose A, B and such that A is a proper subset of B and B is a proper subset of S. Every possible such subset B of size $k < n$ can be chosen in $\binom{n}{k}$ ways, and there is no valid subset of size $k = n$. For each such choice of B of size k, the number of proper subsets A is $2^k - 1$. Hence, we have the total ways are

$$\begin{aligned}
&\sum_{k<n}\binom{n}{k}(2^k-1) \\
=&\sum_{k\le n}\binom{n}{k}(2^k-1) - (2^n-1) \\
=&(3^n - 2^n) - (2^n - 1) = 3^n - 2^{n+1} + 1.
\end{aligned}$$

Note that we used the fact that $\sum_{k\le n}\binom{n}{k}2^k = 3^n$ which we derived in the first part.

g. *Determine a generating function for the series:* $\{0, F_2, 0, F_4, 0, F_6, 0, F_8, \cdots\}$*, where* F_i *represents the* i*'th Fibonacci number.*

Let $f(x) = \sum_{i=0}^{\infty} F_i\, x^i$ be the generating function for the Fibonacci numbers. From class, remember this is

$$f(x) = \frac{x}{1-x-x^2} = \sum_{i\ge 0} F_i x^i.$$

Consider $f(-x)$:

$$f(-x) \;=\; \sum_{i=1}^{\infty} F_i\,(-x)^i \;=\; \sum_{i=1}^{\infty}(-1)^i F_i\, x^i$$

Each term in the power series expansion of $f(-x)$ has the same magnitude as the corresponding term in the expansion of $f(x)$, but the terms with odd i have opposite sign: i.e., $f(-x)$ is the generating function for $F_0, -F_1, F_2, -F_3, \ldots$. Hence if we take the average of the two generating functions, the odd terms are all zero, so the generating function for $F_0, 0, F_2, 0, F_4, 0, \ldots$ is

$$\frac{f(x)+f(-x)}{2} = \frac{1}{2}\frac{x}{1-x-x^2} + \frac{1}{2}\frac{-x}{1+x-x^2}$$

Since $F_0 = 0$, this is the generating function for $0, 0, F_2, 0, F_4, 0, \ldots$. And to shift to the left, we divide out by x, giving us the GF for $0, F_2, 0, F_4, \ldots$ is

$$\frac{1}{2(1-x-x^2)} + \frac{-1}{2(1+x-x^2)}$$