15-251: Great Theoretical Ideas In Computer Science

Homework 3 (due Thursday, September 17)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

0. Warmup (0 points)

Warmup problems need not be turned in. Please solve these and verify your solutions against the sample solutions.

a. What is the probability that a randomly drawn five-card hand will be a full house (a hand with three cards of the same rank, and two cards of a different rank)?

b. How many integers between 1 and 1,000,000 (including both endpoints) are there which are neither of the form x^2 nor x^3, for any natural number x?

c. Convert 11101110010011_2 into decimal; convert 7365350_{10} into binary.

1. Fashion Sense (10 points)

Professor Gupta has four shirts: red, blue, green, and orange. He also has brown, navy, and black pants; pink, yellow, and silver ties; and white and purple shoes. Professor Lafferty has blue, green, and tie-dye shirts; khaki, navy, and teal pants; glow-in-the-dark, pink, yellow, and silver ties; and white and gray shoes. Each of them picks an outfit in the morning, which consists of a shirt, pants, shoes, and *optionally* a tie. However, Profesor Gupta knows that his brown pants do not go with his purple shoes, and that his pink tie does not go with the orange shirt. Professor Lafferty will not wear his glow-in-the-dark tie unless he is also wearing either the tie-dye shirt or the teal pants. Additionally, Professor Gupta may show up with a chef's hat and apron. How many different ways can each professor choose an outfit?

2. Dot Proof (15 points)

Recall the formula for the sum of the first n squares:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use a dot proof to derive this formula. Do not prove this by induction, though you can use induction to prove any supporting claims if needed.

3. Fraternity Roommates (15 points)

a. *(5 points)* There are 16 brothers in the Gamma Nu Upsilon (GNU) fraternity. Their house has 2 identical floors, each of which has 4 rooms, facing North, South, East, and West respectively. Each room houses two brothers, who never leave their room. Each brother can tell which direction his room faces, and who his roommate is, but since the floors are identical, he cannot tell which floor he lives on. How many distinct ways are there to assign brothers to rooms? *(For clarity, an assignment obtained by swapping the residents of one direction on the first floor with the residents of the same direction on the second floor is not considered a different assignment.)*

b. *(10 points)* After a successful recruitment, GNU has n members, and has to move to a larger house. The new house has m rooms on each floor, each of which houses k brothers. Now how many ways are there to assign brothers to rooms? (You may assume n, m, and k are powers of 2, and that $k < n$, but not that $2 * k * m = n$; there may be empty spots or homeless brothers.)

4. GNU Year's Party (35 points)

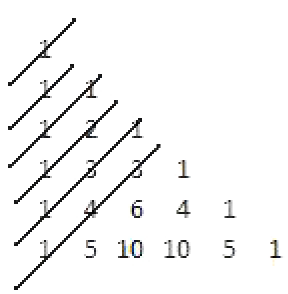
a. *(5 points)* The brothers of GNU have decided to forego their annual new year's dance in favor of a LAN party in formal attire with the sisters of the Beta Sigma Delta sorority. Unfortunately, the only record they have of their membership is in an inconvenient form. There are $X = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$ brothers and $Y = \binom{m+n}{k}$ sisters. Prove by counting in two ways that each brother can escort a distinct date.

b. *(15 points)* At the LAN party, each of the n brothers and n sisters will set up his or her computer on one side of a long table, with n chairs on both sides. Each brother wants to sit either directly across from his date, or immediately adjacent to her. To accomplish this, the party host makes a pair of index cards joined by a short string—just long enough to reach adjacent or opposite chairs —for each couple. He marks one card in each pair "GNU" and the other one "BSD". How many ways are there for him to arrange these cards on the table such that if each brother takes a seat marked "GNU" and his date takes the corresponding card, it will produce a satisfactory arrangement?

c. *(15 points)* How many ways are there to lay out the cards if the LAN party rules additionally require that each brother is seated either next to or across from his date, and also either next to or across from at least one other brother?

5. Climbing Pascal's Triangle (10 points)

The sum of the entries in the n^{th} ascending diagonal of Pascal's triangle, when it is left-justified, is given by $\sum_{i=0}^{n} \binom{n-i}{i}$.



Prove that the sum of these entries is also equal to the n^{th} Fibonacci number, F_n. This gives a combinatorial proof that $F_n = \sum_{i=0}^{n} \binom{n-i}{i}$.

6. Fidocimal Numbers (15 points)

Fido was Fibonacci's faithful dog. Following in his master's footsteps, he created the *Fido* sequence: $D_0 = 2$, $D_1 = 1$, and $D_n = D_{n-1} + D_{n-2}$ for $n > 1$.

In this problem, we consider a number base system built on the Fido numbers. A Fidocimal number consists of a sequence of ones and zeroes $A_n A_{n-1} ... A_1 A_0$, with the numeric value represented being $\sum_{i=0}^{n} A_i \times D_i$, the sum of the Fido numbers with a 1 in the corresponding position. For example, the string "11011" represents $D_4 + D_3 + D_1 + D_0 = 14$. Note, though, that representations in Fidocimal are not unique: "100100" is $D_5 + D_2 = 14$ as well.

To clarify the situation, Fido wrote a paper defining "tidy" Fidocimal numbers as those which do not contain consecutive ones, and do not end in the sequence "0101". He proved that each natural number could be uniquely represented as a *tidy* Fidocimal number. Unfortunately, he ate the proof before it was published. Your task is to prove that each natural number can be represented, and can be represented uniquely, as a tidy Fidocimal number.