15-251: Great Theoretical Ideas In Computer Science

Homework 2 Warmup Solutions

Two players decide to play a game of Nim, but instead of discarding the chips they remove, they keep them. When a player moves, he may either remove chips from a single pile as in regular Nim, or he may add some of the chips he has to a single pile. Analyze this game: what is the terminal position? What are the P- and N-positions?

The terminal position is the one in which there are no chips left in any of the piles, and the current player has no chips saved. Then, the current player can neither add nor remove chips, and he loses the game.

Claim: just as in regular Nim, the P-positions are exactly the ones in which the Nim-sum is zero, and the N-positions are exactly the ones in which the Nim-sum is non-zero. This is proven using the definition of P- and N-positions. The proof is given below.

- First, the terminal position is, by definition, a P-position, and the Nim-sum is indeed zero in this position, so the claim holds here.
- Now, take any position in which the Nim-sum is non-zero. The goal is to prove that this is an N-position.

  The current player can make the Nim-sum zero by *removing* chips from a single pile, just as in regular Nim (and by the same argument). This is a move to an P-position, so the player is currently in an N-position by the definition of N-positions.
- Take any position in which the Nim-sum is zero. The goal is to prove that this is a P-position.

  Regardless of whether the current player adds chips to or removes chips from a single pile, the Nim-sum in the next position will be non-zero, by the same argument as in the proof for regular Nim. Therefore, the current player can only move to an N-position, so, by definition he is currently in a P-position.

The proof also shows that the winning player is able to play only *removing* chips from the game (recall that the winning player is the current player in N-positions). This means that the total number of chips outside his private pile decreases whenever he moves, eventually reaching zero. At this point, the game terminates with him the winner.