15-251: Great Theoretical Ideas In Computer Science

Homework 1 (due Thursday, September 3)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

0. Warmup (0 points)

Warmup problems need not be turned in. Please solve these and verify your solutions against the sample solutions.

a. Prove by induction that $n^n > n!$ for all natural numbers $n > 1$.

b. Give a closed form for the sum $\sum_{k=0}^{n} k \cdot 2^k$. Prove it by induction.

c. What is the lowest integer n_0 such that you can make every amount of postage $n \geq n_0$ cents using only 4-cent, 5-cent and 7-cent stamps? Prove your answer using induction.

d. Show that $7^n + 13^n$ is a multiple of 5 for all odd $n \geq 0$.

1. Dainty Handshaking (20 points)

The Duke of Pumpernickel, pleased with a party he held six months ago, decides to host a handshaking party consisting of a total of n nobles. Unfortunately, the duke was again too enthusiastic with his invitations, and may have accidentally invited a serial liar to the party. At the end of the party, each noble tells the duke the number of people he/she shook hands with (other than himself). In other words, the duke gets a sequence of n numbers.

The duke now wishes to determine if the sequence of n numbers he received was possible at all, that is, if there indeed at least one serial liar among the party. He proposes the following algorithm to determine if a sequence d is legal:

a. Sort the sequence d in decreasing order.

b. If the sequence consists of just the number 0, terminate and report that d is legal.

c. Otherwise, remove the first (and largest) element d_1

d. If the sequence has fewer than d_1 nonzero elements remaining, then terminate and report that d is illegal.

e. Otherwise, subtract 1 from the next d_1 largest numbers $d_2, d_3, \cdots, d_{d_1+1}$ to produce a smaller sequence d'.

f. Recursively apply algorithm to d'

Your task is to find whether or not the above algorithm successfully determines whether any sequence d is legal. Prove your answer correct.

2. Panel Game (15 points)

Rudolf plays a computer game that consists of an $n \times n$ square chessboard displayed on a screen, with each cell colored black or white. The game begins with every other cell colored black, like a regular chessboard. Each move consists of Rudolf choosing a rectangle of cells: the color of every cell in this rectangle of cells then gets flipped. The game ends when all of the cells on the board are colored the same. Rudolf wishes to determine the minimum number of moves required to complete the game.

a. *(5 points)* Give an upper bound on the minimum number of moves that Rudolf needs to complete the game. This means that you should give a strategy that Rudolf can use to finish the game.

b. *(10 points)* Give a lower bound on the minimum number of moves that Rudolf needs to complete the game. If your answer is k, this means that you should prove that no matter how Rudolf chooses to play the game, he must make at least k moves. For full credit, your upper and lower bound must be the same.

3. Another Panel Game (15 points)

George plays a computer game similar to the one given in the above problem. This time, each cell in an $n \times n$ board is filled with a distinct number from 1 to n^2. Each move consists of taking a single line segment of cells (a $1 \times i$ or $i \times 1$ rectangle of cells, where $1 \leq i \leq n$) and reversing the numbers in them. The following examples are valid moves on a 3 by 3 board:

3	**2**	**1**
4	5	6
7	8	9

$\rightarrow$

1	**2**	**3**
4	5	6
7	8	9

3	2	1
4	5	6
7	8	9

$\rightarrow$

4	2	1
3	5	6
7	8	9

The goal of the game is order all of the numbers from least to greatest, wrapping around rows as follows:

1	2	3
4	5	6
7	8	9

George wishes to determine the minimum number of moves M_n required to order the numbers, for the worst-case starting arrangement of the numbers.

a. *(7 points)* Give the best upper bound you can on this quantity M_n.

b. *(8 points)* Give the best lower bound you can for M_n.

Note that in the final answer, your solution for the lower bound should not be a constant.

4. Fun with Identities (20 points)

a. *(5 points)* Give a closed form solution for $f(n)$: $f(0) = 1$, and for all $n \in \mathbb{N}$,

$$f(n+1) = 1 + \sum_{i=0}^{n} f(i).$$

Prove your solution correct.

b. *(7 points)* Let $a_1 = 1$, $a_2 = 2$, and $\forall k \geq 3$, $a_k = 2a_{k-1} - a_{k-2}$. Prove that $\forall n \geq 1$, $a_n \geq n - 1$.

c. *(8 points)* Prove that $(1+x)^n > nx$, for all $x > -1$ and $n \in \mathbb{N}$.

5. Treasure Island (15 points)

A locked treasure box owned by a gang of pirates is located at the center of a vast uninhabited island. The pirates, wanting to keep their wealth protected, erect n fences (where $n \in \mathbb{N} \backslash 0$), each with a single pirate on top of a watch tower on the fence, around the treasure box in order to protect the entire box from being stolen. Unfortunately, although the pirates know that the treasure chest has infinite wealth, they lack the key to open the box. One day, while journeying around the various surrounding islands, you mistakenly find the key to the treasure box. Knowing that the box has infinite wealth, you decide to infiltrate the inhabited island and steal the gems from the treasure box. You decide to strike a deal with each pirate, telling each pirate, "If you let me through, I will give you $2/3$ of the gems that I have when I come back. You must, however, give me one of those gems back". Each pirate accepts, and when you reach the treasure box, you find an almost-infinite number of gems.

a. *(8 points)* This scenario raises the conundrum of splitting gems properly. As a gem loses its worth when cut into pieces, each pirate demands that he/she receives an integer number of gems. Furthermore, each pirate requires at least one gem to be satisfied. Failure to obey these demands, according to the pirates, will result in walking the plank. Give a closed form function $f(n)$ which represents the fewest number of gems that you must bring back from the treasure box to please the pirates and escape the uninhabited island with your life. Prove it.

b. *(7 points)* Suppose that you now give each pirate $1/3$ of the gems when you return to each watchpost (and each pirate still gives one of those gems back). Again, each pirate must receive an integer number of gems greater than 0. Give a closed form function $g(n)$ which represents the fewest number of gems that you must now bring back from treasure box. Prove it.

6. Bad Proofs (15 points)

Your professors need your help in finding any errors in several supposed proofs. Determine whether each of the following proofs is correct. If you believe a proof is correct, say so. If not, then be sure to state, as clearly as you can, exactly what is wrong with the proof, or your answer will be of no use to them! *Note that your professors already know which claims are true. Be sure that you are considering the proofs, not the claims!*

a. *(5 points)* Claim:

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{(n-1)\times n} = \frac{3}{2} - \frac{1}{n}$$

Proof by induction on n:

- *Base case:* $n = 1$: $\frac{3}{2} - \frac{1}{n} = \frac{1}{1\times 2}$.
- *Inductive hypothesis:* Suppose the claim holds for some $n \geq 1$.

- *Inductive step:*

$$\begin{aligned}\frac{1}{1\times 2}+\frac{1}{2\times 3}+\ldots+\frac{1}{(n-1)\times n}+\frac{1}{n\times(n+1)} &= \frac{3}{2}-\frac{1}{n}+\frac{1}{n\times(n+1)}\\ &= \frac{3}{2}-\frac{1}{n}+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{3}{2}-\frac{1}{n+1}\end{aligned}$$

  The first equality holds by the inductive hypothesis. The last expression shows that the claim holds for $n+1$. Therefore, the claim holds for all n by induction.

b. *(5 points)* Claim: Any number of people are of the same gender. Proof:

- *Base case:* When $n = 1$, one person can certainly only be one gender.
- *Inductive hypothesis:* Suppose the claim holds for k people; that is, if $n \leq k$, all sets of n people are of the same gender
- *Inductive step:* We must now show that for all sets of $k+1$ people. First, we remove one person. The remaining k people are the same gender, by the induction hypothesis. We remove another person, re-adding the first one — the same trait remains. We repeat this until all $k+1$ sets of k people have been shown to have the same gender. It then follows that every person has the same gender as every other person. Thus, all sets of $k+1$ people are of the same gender, and the claim is proven by induction.

c. *(5 points)* We'll show that 251 is a magic number. **Claim:** For all positive integers n, we have $251^{n-1} = 1$.
Proof: If $n = 1$, $251^{n-1} = 251^{1-1} = 251^0 = 1$. And by induction, assuming that the theorem is true for $1, 2, \ldots, n$, we have

$$251^{(n+1)-1} = 251^n = \frac{251^{n-1} \times 251^{n-1}}{251^{n-2}} = \frac{1 \times 1}{1} = 1;$$

so the theorem is true for $n+1$ as well.