15-251: Great Theoretical Ideas In Computer Science

Homework 1 Warmup Solutions

a. *Prove by induction that $n^n > n!$ for all natural $n > 1$.*

- *Base case:* $n = 2$: $2^2 = 4$ and $2! = 2$, so $2^2 > 2!$.
- *Inductive hypothesis:* Suppose the claim holds for some $n > 1$.
- *Inductive step:* Then, consider the case for $n + 1$:

$$\begin{aligned}
(n+1)^{n+1} &= (n+1)\cdot(n+1)^n && \text{[algebra]}\\
&> (n+1)\cdot n^n \\
&> (n+1)\cdot n! && \text{[inductive hypothesis]}\\
&= (n+1)! && \text{[def. of !]}
\end{aligned}$$

b. *Give a closed form for the sum $\sum_{k=0}^{n} k \cdot 2^k$. Prove it correct by induction.*

Claim: $\sum_{k=0}^{n} k \cdot 2^k = (n-1)\cdot 2^{n+1} + 2$. Proof by induction on n:

- *Base case:* $n = 0$: $\sum_{k=0}^{0} k \cdot 2^k = 0 = (0-1)\cdot 2^{0+1} + 2$.
- *Inductive hypothesis:* Suppose the claim holds for some $n \geq 0$.
- *Inductive step:* Then,

$$\begin{aligned}
\sum_{k=0}^{n+1} k\cdot 2^k &= (n+1)\cdot 2^{n+1} + \sum_{k=0}^{n} k\cdot 2^k && \text{[algebra]}\\
&= (n+1)\cdot 2^{n+1} + (n-1)\cdot 2^{n+1} + 2 && \text{[inductive hypothesis]}\\
&= (2n)\cdot 2^{n+1} + 2 && \text{[algebra]}\\
&= n\cdot 2^{n+2} + 2
\end{aligned}$$

c. *What is the lowest integer n_0 such that you can make every amount of postage $n \geq n_0$ cents using only 4-cent, 5-cent, and 7-cent stamps? Prove your answer correct using induction.*

n_0 is 7. Claim: for all $n \geq 7$, n cents of postage can be paid using only 4-cent, 5-cent, and 7-cent stamps. Proof by induction on n:

- *Base cases:* $n = 7$: use one 7-cent stamp. $n = 8$: use two 4-cent stamps. $n = 9$: use a 4-cent stamp and a 5-cent stamp. $n = 10$: use two 5-cent stamps.
- *Inductive hypothesis:* Suppose for some $n \geq 11$, the statement holds of $n - 4$; that is, it is possible to pay $n - 4$ cents of postage using only 4-cent, 5-cent, and 7-cent stamps.
- *Inductive step:* Then, it is possible to pay n cents of postage as well: first, pay $n - 4$ cents of postage using the stamps, then add a 4-cent stamp.

The way induction is done in this problem may appear a little strange. You should convince yourself that this indeed covers every natural number above 7: the first four numbers are covered by the base cases. The proof that the claim holds for 11 relies on the claim for 7, the proof for 12 on the claim for 8, and so on.

d. *Show that $7^n + 13^n$ is a multiple of 5 for all odd $n \geq 0$.*

The statement can be rephrased as $7^{2k+1}+13^{2k+1}$ is a multiple of 5 (abbreviated $5|7^{2k+1}+13^{2k+1}$) for all $k \geq 0$. The proof is by induction on k:

- *Base case:* $k = 0$: $7^{2\cdot 0+1} + 13^{2\cdot 0+1} = 20$, and $5|20$.
- *Inductive hypothesis:* Suppose the claim holds for some $k \geq 0$.
- *Inductive step:* Note that if a number A is a multiple of k, then any multiple of A is a multiple of k. Also, if two numbers A and B are both multiples of k, then $A + B$ is also a multiple of k. Note also that $5|120$. Since $13^2 - 7^2 = 169 - 49 = 120$, $5|13^2 - 7^2$. Then,

$$5|7^{2k+1} + 13^{2k+1} \qquad \text{[inductive hypothesis]}$$
$$5|7^2(7^{2k+1} + 13^{2k+1}) \qquad \text{[multiple]} \qquad (1)$$

$$5|13^2 - 7^2 \qquad \text{[above]}$$
$$5|(13^2 - 7^2)13^{2k+1} \qquad \text{[multiple]} \qquad (2)$$

$$5|7^2(7^{2k+1} + 13^{2k+1}) + (13^2 - 7^2)13^{2k+1} \qquad \text{[1, 2, sum of multiples]}$$
$$5|7^2 7^{2k+1} + 13^2 13^{2k+1} \qquad \text{[algebra]}$$
$$5|7^{2(k+1)+1} + 13^{2(k+1)+1}$$

as required.