Imperative Programming

15-150

Lecture 21: November 20, 2025

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Carnegie Mellon University

Let's first continue with streams

Streams*

* (Note, different from SML's built-in I/O streams.)

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Prime? $\longrightarrow$ primes

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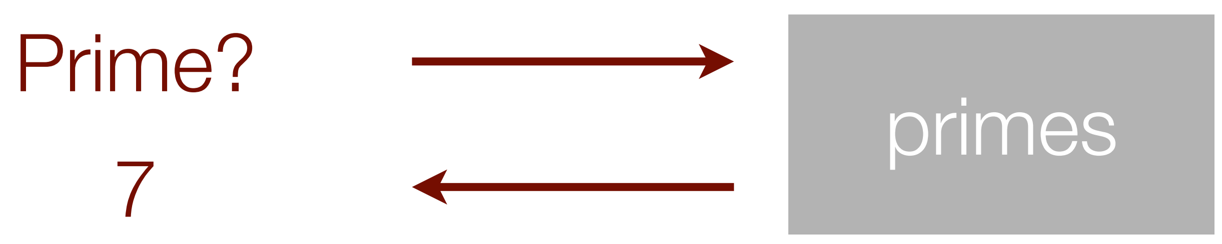

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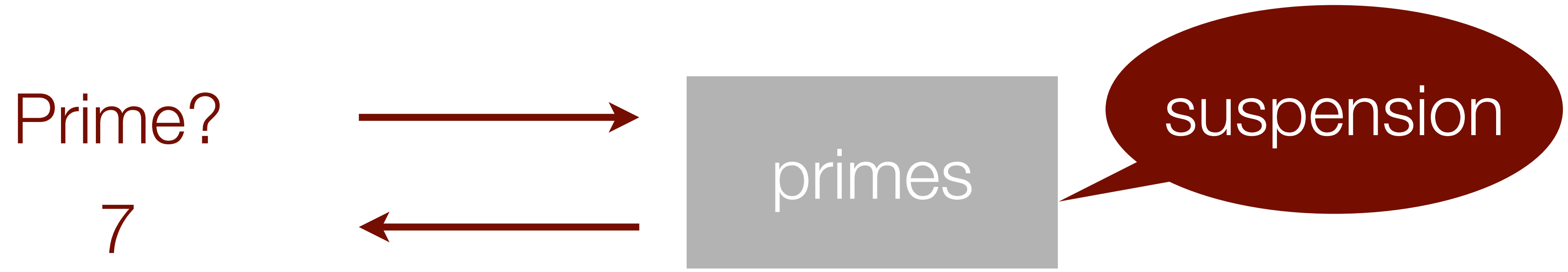

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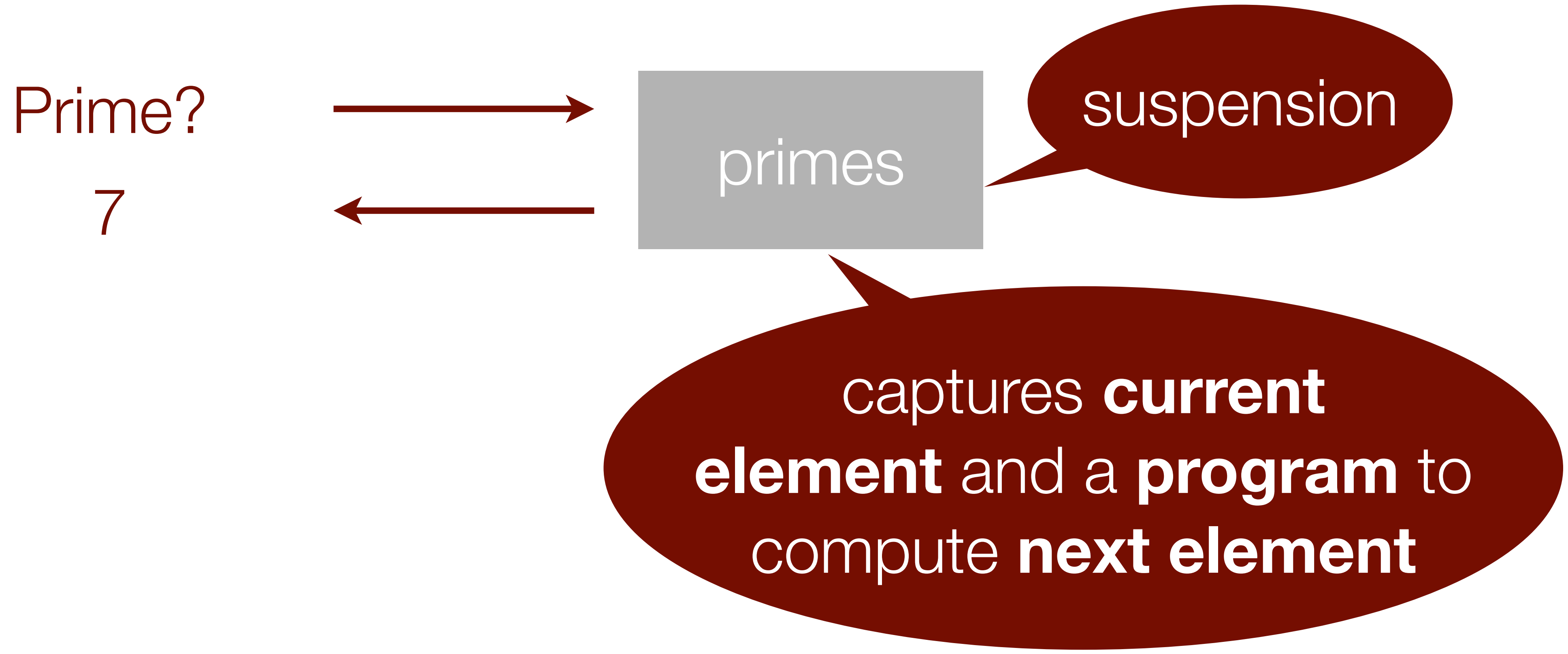

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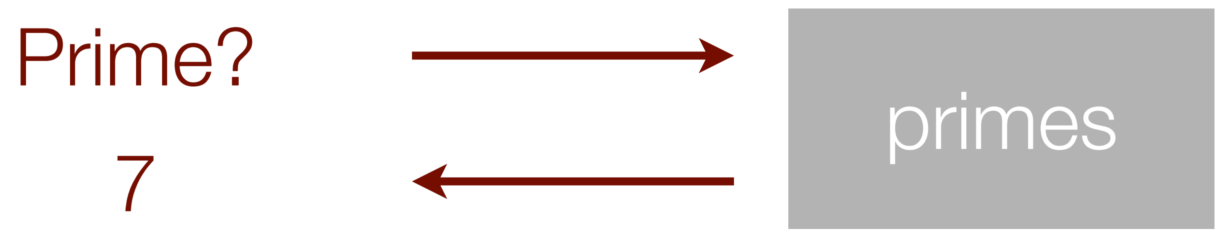

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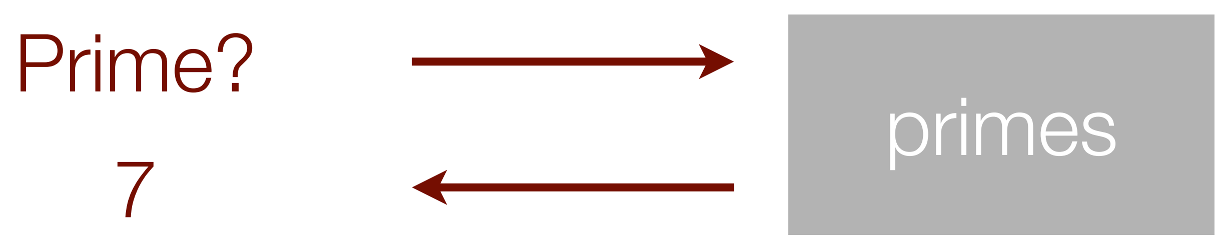

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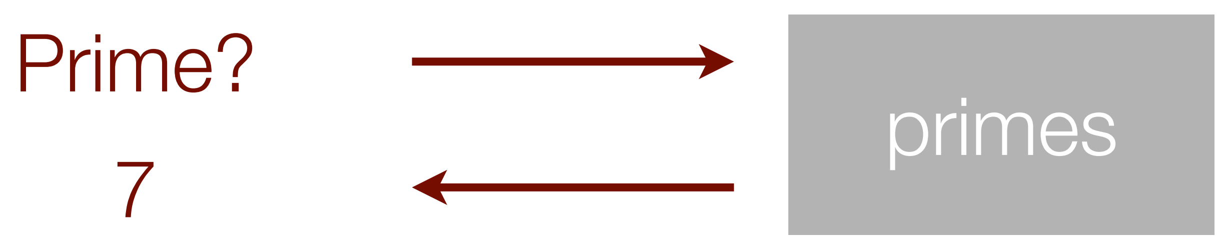

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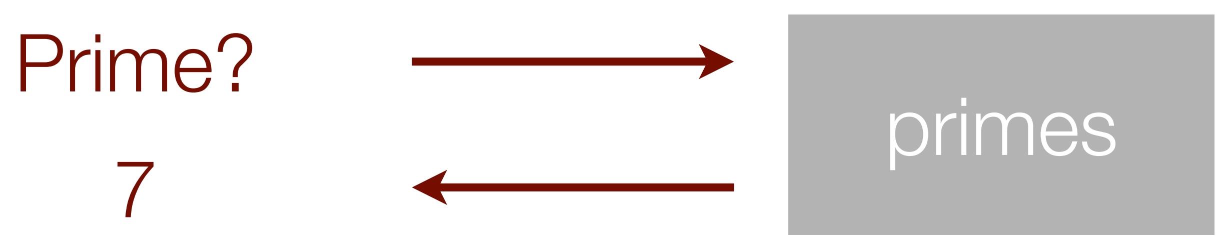

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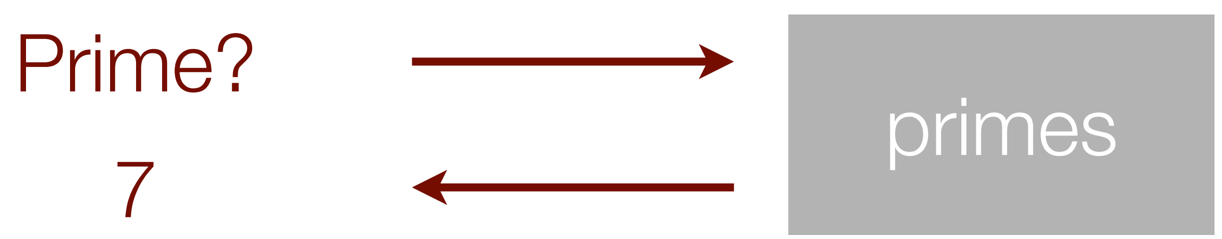

- We can think of streams as being generated by state machines:
  - only when "kicked" (forcing suspension) they yield element
  - advancing state for computation of next element.
- Streams are defined **coinductively**.

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Stream signature

```
signature STREAM =
sig
  type 'a stream                          (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                    | Empty               (* concrete *)

  val expose : 'a stream -> 'a front

  val delay : (unit -> 'a front) -> 'a stream

  (* more functions (see accompanying code) *)
end
```

Stream structure

```
structure Stream : STREAM =
struct
  datatype 'a stream = Stream of unit -> 'a front
  and 'a front = Cons of 'a * 'a stream | Empty

  (* delay : (unit -> 'a front) -> 'a stream *)
  fun delay (d) = Stream(d)

  (* expose : 'a stream -> 'a front *)
  fun expose (Stream(d)) = d ()

  (* more functions (see accompanying code) *)
end
```

Let's practice: stream of nats

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Assume that the following codes is written outside the `Stream` structure, where we abbreviate `Stream` with `S` for space reasons.

Let's implement an infinite stream of all natural numbers:

```
(* nat' : int -> unit -> int S.front *)
fun nat' x () = S.Cons(x, S.delay (nat' (x+1)))

(* int S.stream *)
val nats = S.delay (nat' 0)
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current element

next element

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Another example: prime numbers

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Inspired by the Sieve of Eratosthenes.

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```
2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,...
```

Another example: prime numbers

Inspired by the Sieve of Eratosthenes.

`2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,...`

Write down all the natural numbers greater than `1`.

Another example: prime numbers

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Find leftmost element (`2` currently).

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Cross off all multiples of that leftmost element.

Another example: prime numbers

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`2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~, 17, ~~18~~, ...`

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3, 5, 7, 9, 11, 13, 15, 17,...

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Repeat the process with the remaining numbers.

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Keep repeating this process.

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The diagonal of leftmost elements constitutes all primes.

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delays actual sieving

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not really needed because primes are infinite

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filters multiples of
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recursively constructs stream of larger primes, with p at front

filters multiples of current element p

Recall: `(* delay` … `-> ` … `*)`
`fun delay`

Imperative programming

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- But what does **pure** really mean?
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- But what does that really mean? 🤔

Functional programming

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→ But what does **pure** really mean?

→ Well, the prototypical answer is, **without** any **side-effects**.

Let's reconsider the correctness proofs that we carried out.

Functional programming

So far we have used the term "functional programming" as a synonym for pure programming.

- But what does **pure** really mean?
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- Expressions that engender effects typically are of `unit` type.

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Today's menu

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  - reference type
  - typing and evaluation rules

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Mutable reference cells

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arbitrary SML type*

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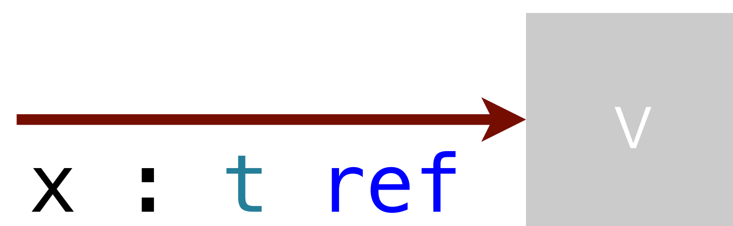

(where `v : t`)

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`x : t ref` → v (where `v : t`)

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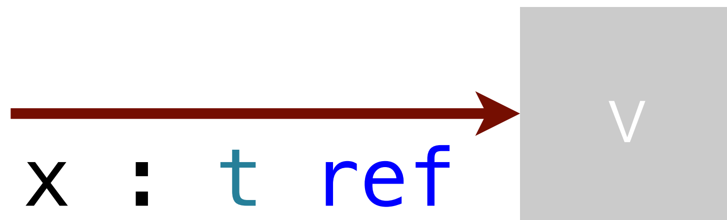

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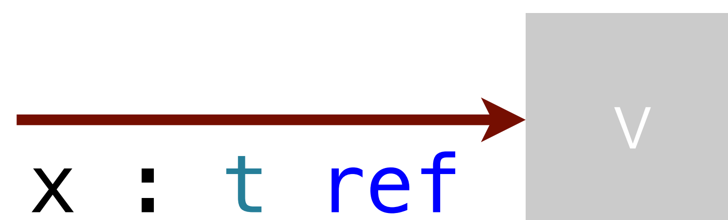

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Functions:	`ref : 'a -> 'a ref`	allocation
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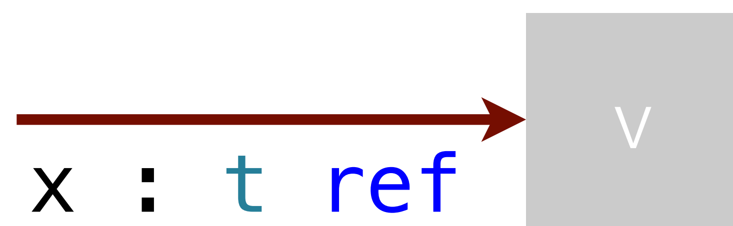

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```
ref : 'a -> 'a ref              allocation
!   : 'a ref -> 'a              read
:=  : 'a ref * 'a -> unit       write
```

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Allocation: `ref: 'a -> 'a ref`

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val r = ref (1 + 3)
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```
val r = ref (1 + 3)
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evaluates to: r → 4

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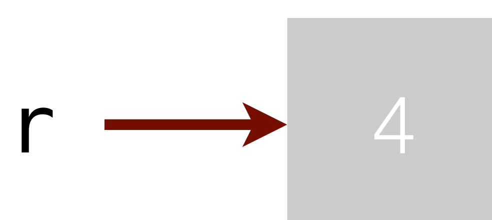

Here, `r : int ref` is bound to a reference to the reference cell containing the value `4 : int`.

Allocation: `ref: 'a -> 'a ref`

Evaluation rules: `ref e`

1. Evaluate expression **e**.

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Evaluation rules: `ref e`

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- If `e : t`, then `ref e : t ref`.

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Example:

```
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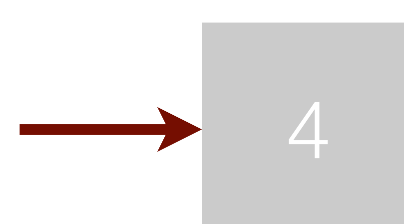
val x = !r
```

evaluates to: r



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Example:

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Example:

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evaluates to: `r` ⟶ 4 and `[4/x]`

Here, `r : int ref` is bound to a reference to the cell containing the value `4 : int` and `x : int` is bound to `4`.

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1. Evaluate expression e_1.
2. If e_1 reduces to a reference `r`, then evaluate expression e_2.
3. If e_2 reduces to a value `v`, update contents of `r` to `v`, return `()`.

Write: `:= : 'a ref * 'a -> unit`

Evaluation rules: `e1 := e2` (e_1 := e_2)

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Example:

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val r = ref (1 + 3)
r := (!r * 2)
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evaluates to: r → 8

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Evaluation rules: `e1 := e2`

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Typing rules: e_1 := e_2

If e_1 `: t ref` and e_2 `: t`, then e_1 `:=` e_2 `: unit`.

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We can pattern match on `ref`:

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(* containsZero : int ref -> bool *)

fun containsZero (ref 0) = true
  | containsZero _ = false
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val false = containsZero (ref 7)
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Reference cells support pattern matching

We can pattern match on `ref`:

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val false = containsZero (ref 7)

val true = containsZeros (ref 0)
```

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1 Evaluate e_1, executing effects but ignoring any returned value.

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$$(e_1;\ e_2)$$

Which is syntactic sugar for:

$$\texttt{let val \_ = } e_1 \texttt{ in } e_2 \texttt{ end}$$

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For convenience, SML supports the semicolon expression:

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1. Evaluate e_1, executing effects but ignoring any returned value.
2. Then, evaluate e_2, executing effects and return the value of e_2.

Generalizes to:

$$(e_1;\ e_2;\ \ldots;\ e_n) : t_n$$

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let
  val c = ref 10
in
  (print(Int.toString(!c));
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What is its value? `ref 10`

What its effect?

Sequential composition

Example:

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let
  val c = ref 10
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   c)
end
```

What is the type of this `let` expression? `int ref`

What is its value? `ref 10`

What its effect? prints 10

Sequential composition

Alternative implementation of previous example:

```
let
  val c = ref 10
  val _ = print(Int.toString(!c))
in
  c
end
```

Aliasing

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```
val c = ref 10
val w = !c
val d = c
val () = d := 42
val v = !c
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`d` is now referring to the same cell as `c`

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evaluation may alter
the store!

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More on this in 15-312!

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- **We won't go into any further details in 15-150.**
  - **More on this in 15-312!**
- **Note: aliasing complicates reasoning about programs 😢**

Extensional equivalence

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Note:

- `ref` types are so called equality types

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Note:

- `ref` types are so called equality types

For `r : 'a ref` and `s : 'a ref`, `r = s` evaluates to `true`, if `r` and `s` are aliases, i.e., point to the same cell.

Race conditions

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fun deposit a n = a := !a + n

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val chk = ref 100

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70
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Now, if we parallelize, what is the value of !`chk`?

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We could end up with `20`, `70`, or `150`.

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Mutation and parallelism leads to non-deterministic outcomes 😢

Persistent versus ephemeral data

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Pure programs:

yield persistent data structures

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Persistent versus ephemeral data

Pure programs:

- yield persistent data structures
- facilitate reasoning and support deterministic parallelism

Imperative programs:

- yield ephemeral data structures
- complicate reasoning and demand concurrent scheduling

However, not all effects are evil.

- When employed locally, effects can be **benign**.

Benign effects

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A **benign effect** is an effect (such as mutation) that is **localized** within some sufficiently small chunk of code (e.g., function or structure) so that external users can use the code as **if it were purely functional**.

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- Because effect is local, local reasoning remains intact.
- Let's look at some examples!

Example: graph reachability

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Consider this directed graph:

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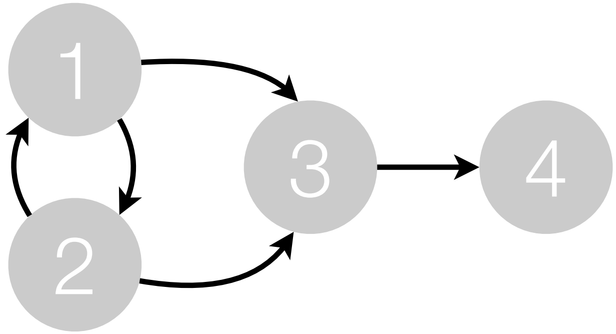

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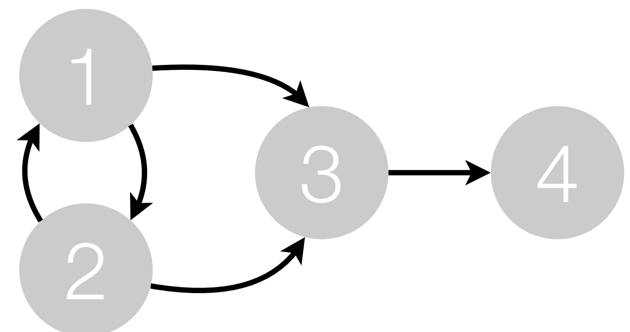

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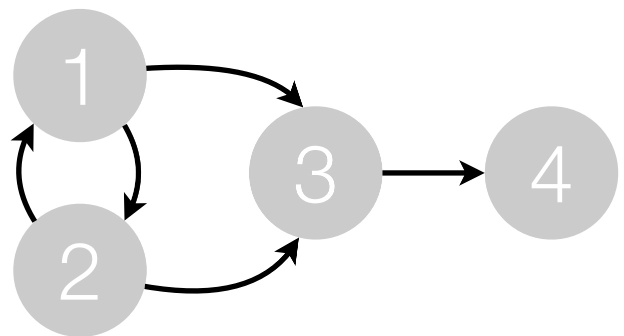

We can represent this graph as a function, giving for a node the nodes immediately reachable from it:

```
type  graph = int -> int list

val G : graph = fn 1 => [2,3]
                 | 2 => [1,3]
                 | 3 => [4]
                 | _ => [ ]
```

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neighbors of **n**

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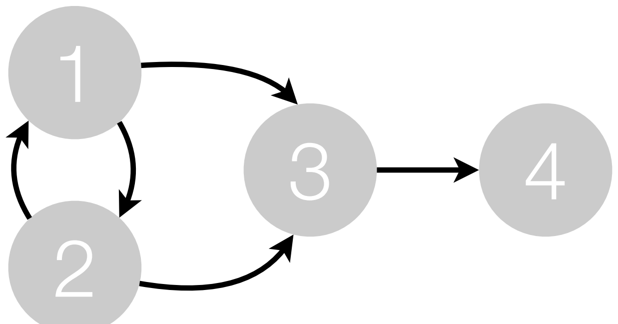

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mem n L checks whether n is in list L

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reference that records visited nodes

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only continue if **n** has not yet been visited

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update visited list

Example: random number generator

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(Reference: "ML for the Working Programmer" by Paulson. 1996.)

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```
signature RANDOM =
sig
  type gen (*abstract *)
  val init: int -> gen (* REQUIRES: seed > 0 *)
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bound

pseudo-random
nonnegative integer
less than bound

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Example: random number generator

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signature RANDOM =
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val G = R.init(12345)
val L = List.tabulate(42,fn _ => R.random G 1000)
```

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struct R :> RANDOM
  type gen = real ref
  val a = 16807.0
  val m = 2147483647.0
  fun next r = a * r - m*real(floor(a*r/m))
  val init = ref o real
  fun random g b = (g := next(!g);
                    floor( (!g/m)* (real b)))
end
```

(Reference: "ML for the Working Programmer" by Paulson. 1996.)

Example: random number generator

```
signature RANDOM =
sig
  type gen (*abstract *)
  val init: int -> gen (* REQUIRES: seed > 0 *)
  val random: gen -> int -> int
end

struct R :> RANDOM
  type gen = real ref
  val a = 16807.0
  val m = 2147483647.0
  fun next r = a * r - m*real(floor(a*r/m))
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reference cell

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Example: stream memoization

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(* delay : (unit -> 'front) -> 'a stream *)
fun delay (d) = Stream(d)

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fun expose (Stream(d)) = d ()
```

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Let's add a hidden reference cell that remembers the result of computing `d()`.

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```

Let's add a hidden reference cell that remembers the result of computing `d()`.

We will will leave `expose` as is, but change `delay`.

Example: stream memoization

Example: stream memoization

Updated function `delay`:

Example: stream memoization

Updated function **`delay`**:

```
fun delay (d) =
  let







  in

  end
```

Example: stream memoization

Updated function **`delay`**:

```
fun delay (d) =
  let
    val cell = ref d









  in

  end
```

Example: stream memoization

Updated function **`delay`**:

```
fun delay (d) =
  let
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  in

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let's put **d** in a reference cell

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fun delay (d) =
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let's put **d** in a reference cell

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Recall:

```
(* expose : 'a stream -> 'a front *)
fun expose (Stream(d)) = d ()
```

Example: stream memoization

Updated function **`delay`**:

```
fun delay (d) =
  let
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Recall:

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(* expose : 'a stream -> 'a front *)
fun expose (Stream(d)) = d ()
```

```
  in
    Stream (fn () => !cell())
  end
```

Example: stream memoization

Updated function **`delay`**:

```
fun delay (d) =
  let
    val cell = ref d
  in
    Stream (fn () => !cell())
  end
```

let's put **d** in a reference cell

Recall:

```
(* expose : 'a stream -> 'a front *)
fun expose (Stream(d)) = d ()
```

we now need a suspension, when forced, accesses the reference cell and forces the function in the reference cell

Example: stream memoization

Updated function **`delay`**:

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fun delay (d) =
  let
    val cell = ref d









  in
    Stream (fn () => !cell())
  end
```

Example: stream memoization

Updated function **`delay`**:

```
fun delay (d) =
  let
    val cell = ref d
    fun memoFn () =
      let
        val r = d()
      in
        (cell := (fn () => r); r)
      end

  in
    Stream (fn () => !cell())
  end
```

Example: stream memoization

Updated function **`delay`**:

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```

`memoFn` is a function that computes `d()`, remembers the result `r` in a suspension, puts that suspension in `cell`, and returns `r`.

Example: stream memoization

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    val cell = ref d
    fun memoFn () =
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```

Example: stream memoization

Updated function **`delay`**:

```
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  let
    val cell = ref d
    fun memoFn () =
      let
        val r = d()
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        (cell := (fn () => r); r)
      end
    val _ = cell := memoFn
  in
    Stream (fn () => !cell())
  end
```

Example: stream memoization

Updated function **`delay`**:

```
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  let
    val cell = ref d
    fun memoFn () =
      let
        val r = d()
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        (cell := (fn () => r); r)
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    val _ = cell := memoFn
  in
    Stream (fn () => !cell())
  end
```

we put **`memoFn`** into **`cell`**, where it will sit until someone exposes the stream, at which point **`memoFn`** replaces itself with **`fn () => r`**.

That's all for today. Have a good weekend!