

Lazy Programming

15-150

Lecture 20: November 18, 2025

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Carnegie Mellon University

Today

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- Keystrokes made on a keyboard
- My email inbox (😄)
- Video / audio streams

➔ To program infinite data structures, we use the notion of a **delayed computation**.

➔ Delayed computations also facilitate **demand-driven** (aka **lazy**) programming in a call-by-value language such as SML.

Delayed computation

Delayed computation

Idea:

Delayed computation

Idea:

→ Encapsulate computation to **suspend** it.

Delayed computation

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- ➔ Encapsulate computation to **suspend** it.
- ➔ Execute computation by explicitly **forcing** it.

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Let's take a step back and ask ourselves the following question:

What is the difference between the following two expressions?

`e`

and

`fn x => e x`

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Here, SML will evaluate `e`.

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What is the difference between the following two expressions?

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Here, SML will evaluate `e`.

and

`fn x => e x`

Here, SML will only evaluate `e`, when the lambda is applied to an argument.

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Idea:

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➔ Lambdas allow us to suspend computations!

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➔ Lambdas are values (even if encapsulated computation diverges).

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For example, given

`fun g x = g x`

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and

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```

```
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and

```
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Idea:

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Can we do that in SML? 🤔

For example, given

```
fun g x = g x
```

`g 3` loops, but `fn x => (g 3) x` is a value

➔ Lambdas allow us to suspend computations!

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Delayed computation

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➔ Yes, using lambdas to represent infinite, possibly diverging computations.

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- ➔ We call such lambdas **suspensions**:

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A **suspension** of type τ is a function f of type

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A **suspension** of type t is a function f of type

$$f: \text{unit} \rightarrow t$$

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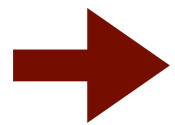
such that for $e: t$, f is `fn () => e`.

Delayed computation

A **suspension** of type t is a function f of type

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such that for $e: t$, f is $\text{fn } () \Rightarrow e$.



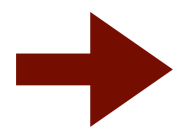
A suspension is **forced**, when it is applied, i.e., $f ()$.

Delayed computation

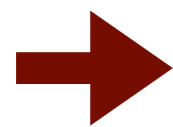
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The suspension f is a **lazy** representation of e because e won't be evaluated until f is forced.

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- ➔ A suspension is **forced**, when it is applied, i.e., $f ()$.
- ➔ The suspension f is a **lazy** representation of e because e won't be evaluated until f is forced.
- ➔ Let's use suspensions to represent (possibly infinite) **streams** of data.

Streams*

* (Note, different from SML's built-in I/O streams.)

Streams*

Streams are data structures that are being continuously created, e.g.,

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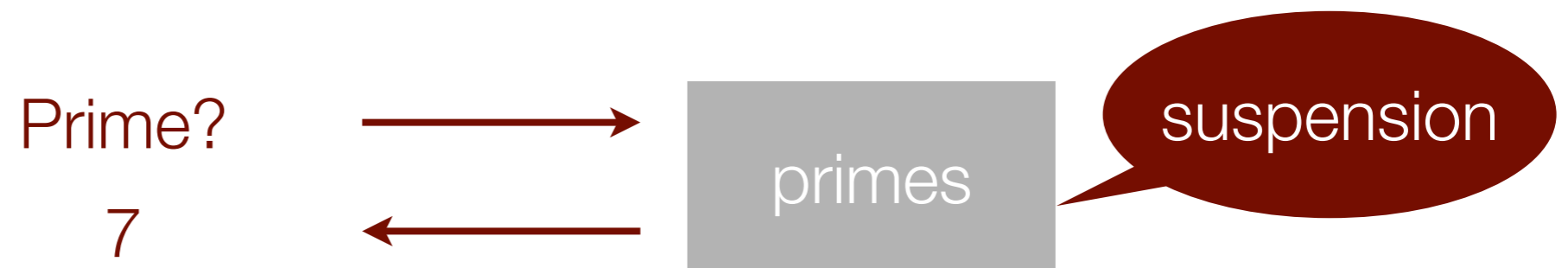
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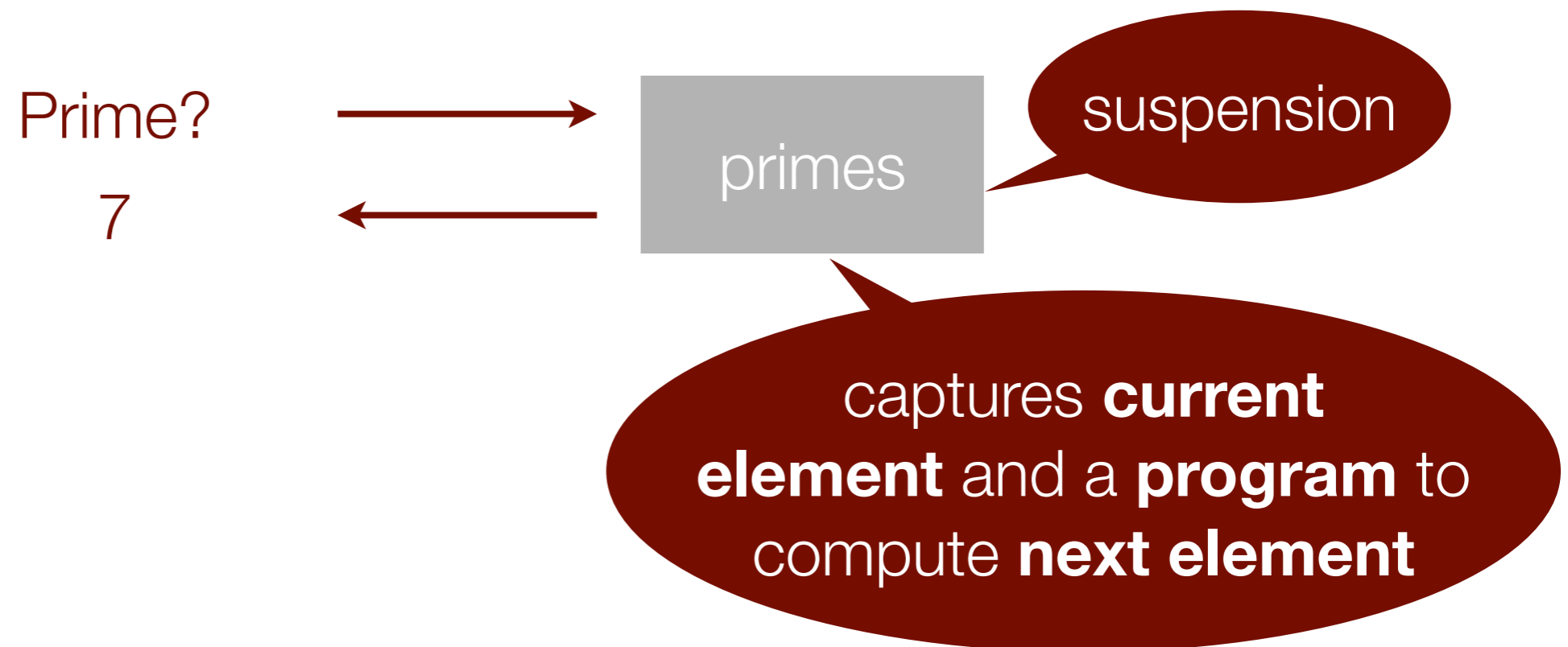
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- ➔ We can think of streams as being generated by state machines:
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- ➔ We can think of streams as being generated by state machines:
- ➔ only when "kicked" (forcing suspension) they yield element
- ➔ advancing state for computation of next element.
- ➔ Streams are defined **coinductively**.

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Intermezzo: induction versus coinduction

Intermezzo: induction versus coinduction

Intermezzo: induction versus coinduction

if you'd like to
know

Intermezzo: induction versus coinduction

if you'd like to
know

aka, we
don't expect you to
know

Intermezzo: induction versus coinduction

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➔ show containment of element by consistent behavior

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→ Inductive data types facilitate **proofs by induction**

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→ Coinductive data types facilitate **proofs by coinduction**

→ show containment of element by consistent behavior

Intermezzo: induction

We can also define corresponding lazy versions!

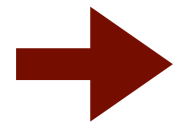
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 - show that all possible ways of construction satisfy property
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 - show containment of element by consistent behavior

Let's implement streams

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First, we define a signature, capturing streams abstractly.

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- ➔ First, we define a signature, capturing streams abstractly.
- ➔ Then, we implement them in a corresponding structure.

Stream signature

Stream signature

```
signature STREAM =
```

```
sig
```

```
  type 'a stream
```

```
  (* abstract *)
```

```
end
```

Stream signature

```
signature STREAM =
```

```
sig
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```
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```
(* abstract *)
```



streams with
elements of type 'a

```
end
```


Stream signature

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Stream signature

```
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```

```
  type 'a stream                                (* abstract *)
```

```
  datatype 'a front = Cons of 'a * 'a stream    (* concrete *)
```

```
end
```

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➔ Forcing ("kicking") a stream yields a value of type 'a front,

```
end
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  type 'a stream                                (* abstract *)
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```

➔ Forcing ("kicking") a stream yields a value of type 'a front,

➔ comprising the current element

```
end
```

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  type 'a stream                                (* abstract *)
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  datatype 'a front = Cons of 'a * 'a stream    (* concrete *)
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➔ Forcing ("kicking") a stream yields a value of type 'a front,

➔ comprising the current element and the rest of the stream,

```
end
```

Stream signature

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signature STREAM =
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```
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```

```
  type 'a stream                                (* abstract *)
```

```
  datatype 'a front = Cons of 'a * 'a stream  
                    | Empty      (* concrete *)
```

➔ Forcing ("kicking") a stream yields a value of type 'a front,

➔ comprising the current element and the rest of the stream,

➔ or Empty, in case the stream is finite.

```
end
```

Stream signature

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  type 'a stream                                (* abstract *)
```

```
  datatype 'a front = Cons of 'a * 'a stream  
                    | Empty      (* concrete *)
```

```
end
```


Stream signature

```
signature STREAM =
sig
  type 'a stream                                (* abstract *)

  datatype 'a front = Cons of 'a * 'a stream
                    | Empty                       (* concrete *)

  val expose : 'a stream -> 'a front
end
```

Stream signature

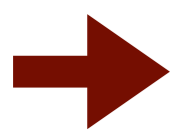
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Function `expose` forces the computation yielding the current element and the remainder of the stream.

```
end
```

Stream signature

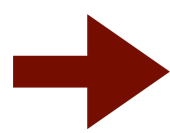
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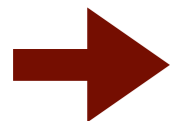
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  type 'a stream                                (* abstract *)
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```

```
  val expose : 'a stream -> 'a front
```



Function `expose` forces the computation yielding the current element and the remainder of the stream.



Caution: `expose` may loop!

```
end
```

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  val expose : 'a stream -> 'a front

  val delay : (unit -> 'a front) -> 'a stream

end
```

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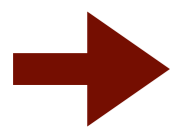
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Function `delay` creates a stream, given a suspension for computing the stream.

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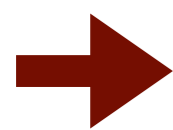
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Function `delay` creates a stream, given a suspension for computing the stream.



Suspension required, otherwise SML will evaluate argument!

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Stream signature

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  datatype 'a front = Cons of 'a * 'a stream
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  val expose : 'a stream -> 'a front

  val delay : (unit -> 'a front) -> 'a stream

  (* more functions (see accompanying code) *)
end
```

Stream structure

Stream structure

```
structure Stream : STREAM =  
struct  
  datatype 'a stream = Stream of unit -> 'a front
```

```
end
```

Stream structure

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```
struct
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  datatype 'a stream = Stream of unit -> 'a front
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end
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sig  
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  ...  
end
```

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```
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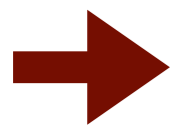
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  ...  
end
```

Stream structure

```
structure Stream : STREAM =
```

```
struct
```

```
  datatype 'a stream = Stream of unit -> 'a front
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We find it convenient to wrap a `Stream` constructor around the suspension of an `'a front`.

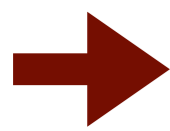
```
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```

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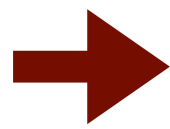
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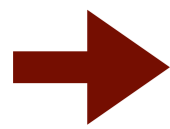
```
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  ...  
end
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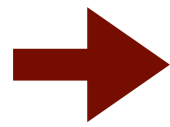
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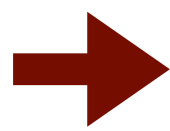
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```

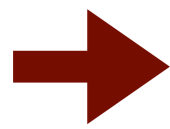

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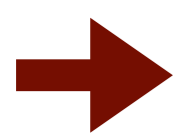
```
end
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end
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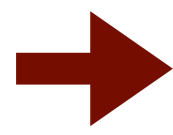
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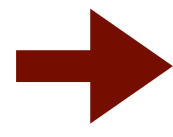
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  datatype 'a stream = Stream of unit -> 'a front
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Recall: `'a front` refers to `'a stream`.

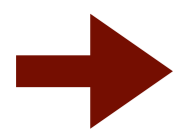
```
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```

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```

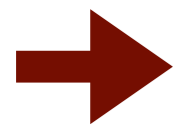
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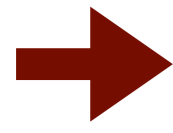
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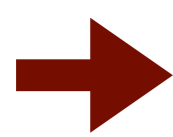
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```
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```

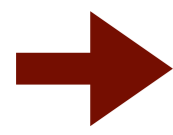
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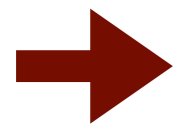
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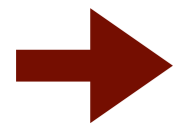
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The use of the constructor `Stream`, instead of the plain suspension, conveys more readily what the function is about.



Recall: `'a front` refers to `'a stream`.



How do we handle that?

```
end
```

Stream structure

```
structure Stream : STREAM =
```

```
struct
```

```
  datatype 'a stream = Stream of unit -> 'a front
```

```
  and 'a front = Cons of 'a * 'a stream | Empty
```

```
end
```

```
signature STREAM =
```

```
sig
```

```
  type 'a stream
```

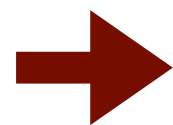
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```
  ...
```

```
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```

Stream structure

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Define mutually recursive data structures with keyword `and`.

```
end
```

```
signature STREAM =  
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Stream structure

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```

➔ Define mutually recursive data structures with keyword `and`.

➔ Recall: `'a front` is already defined as such in signature.

```
end
```

```
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  ...  
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Stream structure

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  fun delay (d) = Stream(d)
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➔ Wraps Stream constructor around suspension of 'a front.

```
end
```

Stream structure

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  fun delay (d) = Stream(d)  
  
  (* expose : 'a stream -> 'a front *)  
  fun expose (Stream(d)) = d ()  
  
  → Forces underlying suspension in input stream.  
  
end
```

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  fun delay (d) = Stream(d)

  (* expose : 'a stream -> 'a front *)
  fun expose (Stream(d)) = d ()

  (* more functions (see accompanying code) *)
end
```

Let's practice: stream of 1s

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Assume that the following codes is written outside the **Stream** structure, where we abbreviate **Stream** with **S** for space reasons.

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→ Let's implement an infinite stream whose elements are 1:

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➔ Let's implement an infinite stream whose elements are 1:

```
(* ones' : unit -> int S.front *)  
fun ones' () = S.Cons(1, S.delay ones')  
  
(* int S.stream *)  
val ones = S.delay ones'
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current element

remains the same in tail

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```

current element

next element

```
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Let's practice: stream of nats

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Consider now:

Let's practice: stream of nats

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Consider now:

```
val S.Cons(x, tail) = S.expose nats  
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What values are `x` and `y` bound to? What does `tail` represent?

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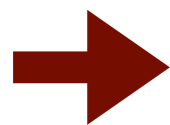
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x is bound to 0 and y to 1

Let's practice: stream of nats

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val S.Cons(x, tail) = S.expose nats  
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What values are **x** and **y** bound to? What does **tail** represent?

➔ x is bound to 0 and y to 1

➔ tail denotes the stream of all natural numbers greater than 0

Let's practice: stream of nats

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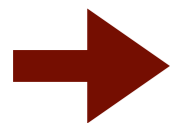
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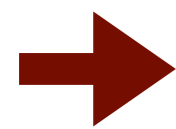
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`take(s, n)` returns the first `n` elements of stream `s` as a list.

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```
Stream.take(X,n)  $\cong$  Stream.take(Y,n)
```


Another example: prime numbers

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2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, . . .

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Write down all the natural numbers greater than **1**.

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Find leftmost element (2 currently).

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Cross off all multiples of that leftmost element.

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Repeat the process with the remaining numbers.

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The diagonal of leftmost elements constitutes all primes.

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Recall: (* delay : (unit -> 'front) -> 'a stream *)
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delays
actual sieving

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fun sieve s = S.delay (fn () => sieve (compose s))
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val primes = sieve (S.delay (nat' 2))
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not really needed
because primes are
infinite

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Recall: (* delay
fun delay

recursively
constructs stream of
larger primes, with p
at front

filters multiples of
current element p

That's all for today.