15-150

Principles of Functional Programming

Slides for Lecture 18

Red Black Trees

March 26, 2024

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Main Lesson:

- How to maintain Representation Invariants within a structure when some code necessarily violates the invariants:
  - Localize the violation
  - Characterize the violation
  - Determine weakened invariants
  - Write code that re-establishes the original invariants from data satisfying the weakened invariants

Ancillary Lessons:

- Functional implementation of balanced trees
- Pattern-matching as code-by-picture ;-)

Dictionary Signature

```
signature DICT =
sig
  type key = string                  (* concrete *)
  type 'a entry = key * 'a           (* concrete *)

  type 'a dict                       (* abstract *)

  val empty : 'a dict

  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

Dictionary Implementation

```
signature DICT =
sig
  type key = string                (* concrete *)
  type 'a entry = key * 'a         (* concrete *)

  type 'a dict                     (* abstract *)

  val empty : 'a dict

  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

Last week we implemented

```
structure BinarySearchTree : DICT = ...
```

using a tree to represent a dictionary, with the Representation Invariant that the tree is **sorted** on `key` (and there are no duplicate keys).

Red Black Tree Dictionaries

Binary search tree with Red and Black nodes:

```
datatype 'a dict =
   Empty
 | Red of 'a dict * 'a entry * 'a dict
 | Black of 'a dict * 'a entry * 'a dict
```

(`Empty` considered black.)

Red Black Tree Dictionaries

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datatype 'a dict =
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(`Empty` considered black.)

Red Black Tree (RBT) Invariants:

(1) The tree is **sorted** on the `key` part of the entries.

(2) The **children** of a `Red` node are `Black`.

(3) Each node has a well-defined *black height*:
The number of `Black` nodes on *any* path
from the node down to an `Empty` is the same.

Red Black Tree Dictionaries

Binary search tree with Red and Black nodes:

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(`Empty` considered black.)

Red Black Tree (RBT) Invariants:

(1) The tree is **sorted** on the `key` part of the entries.

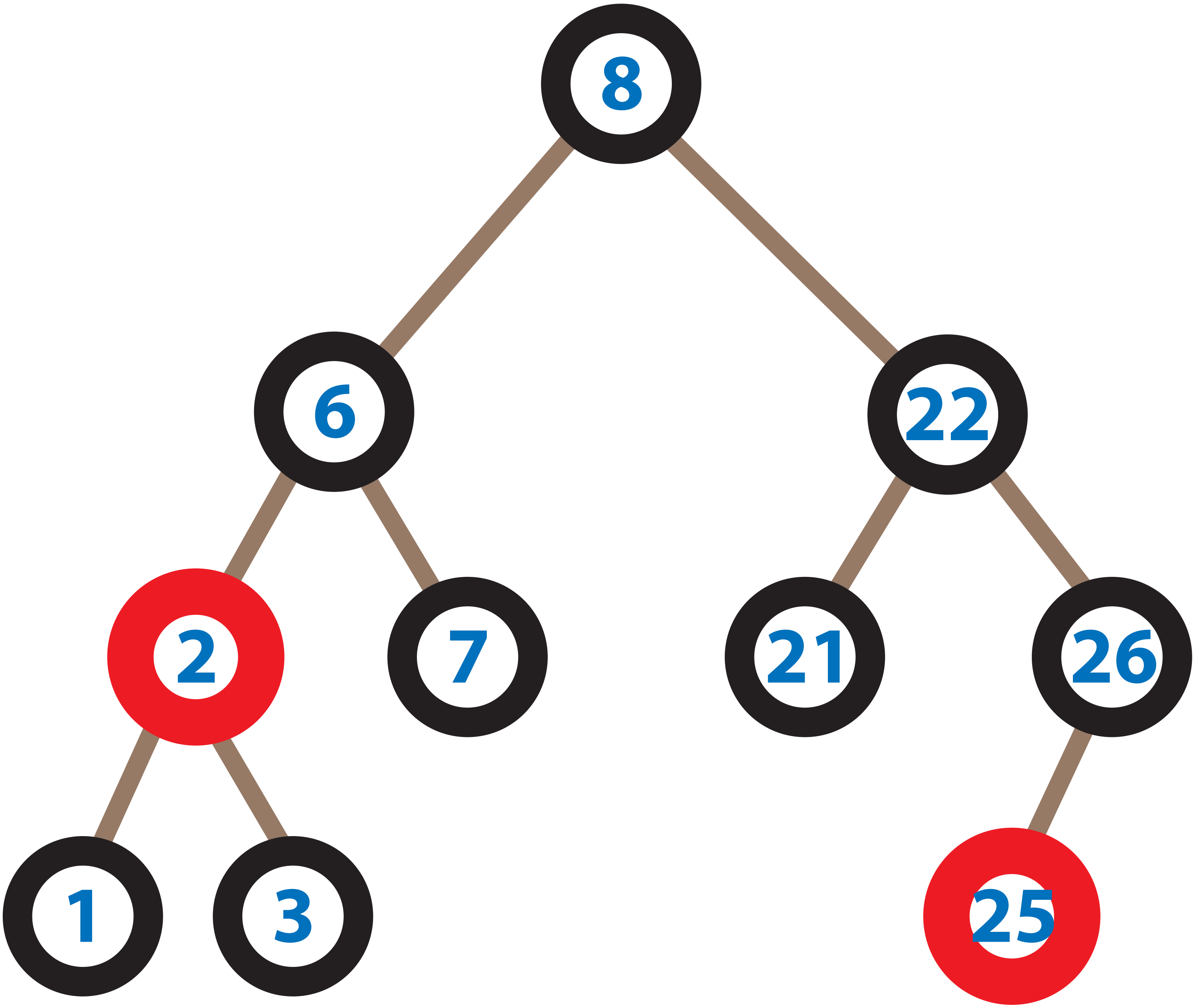
(2) The **children** of a `Red` node are `Black`.

(3) Each node has a well-defined *black height*:
The number of `Black` nodes on *any* path from the node down to an `Empty` is the same.

Invariants imply the tree is roughly balanced:

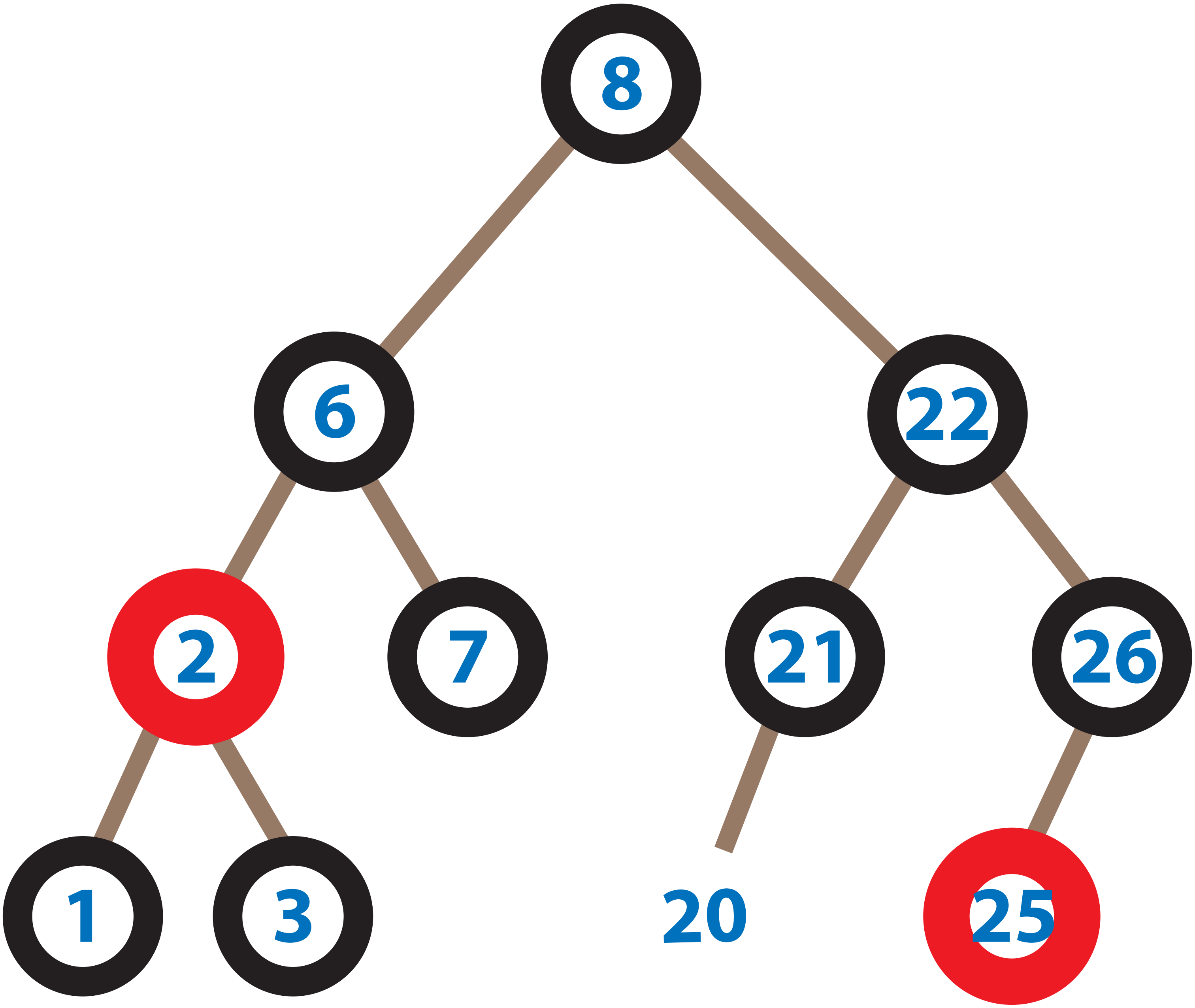
$$\text{depth} \le 2\log_2(|\text{nodes}| + 1)$$

A given Red Black Tree:

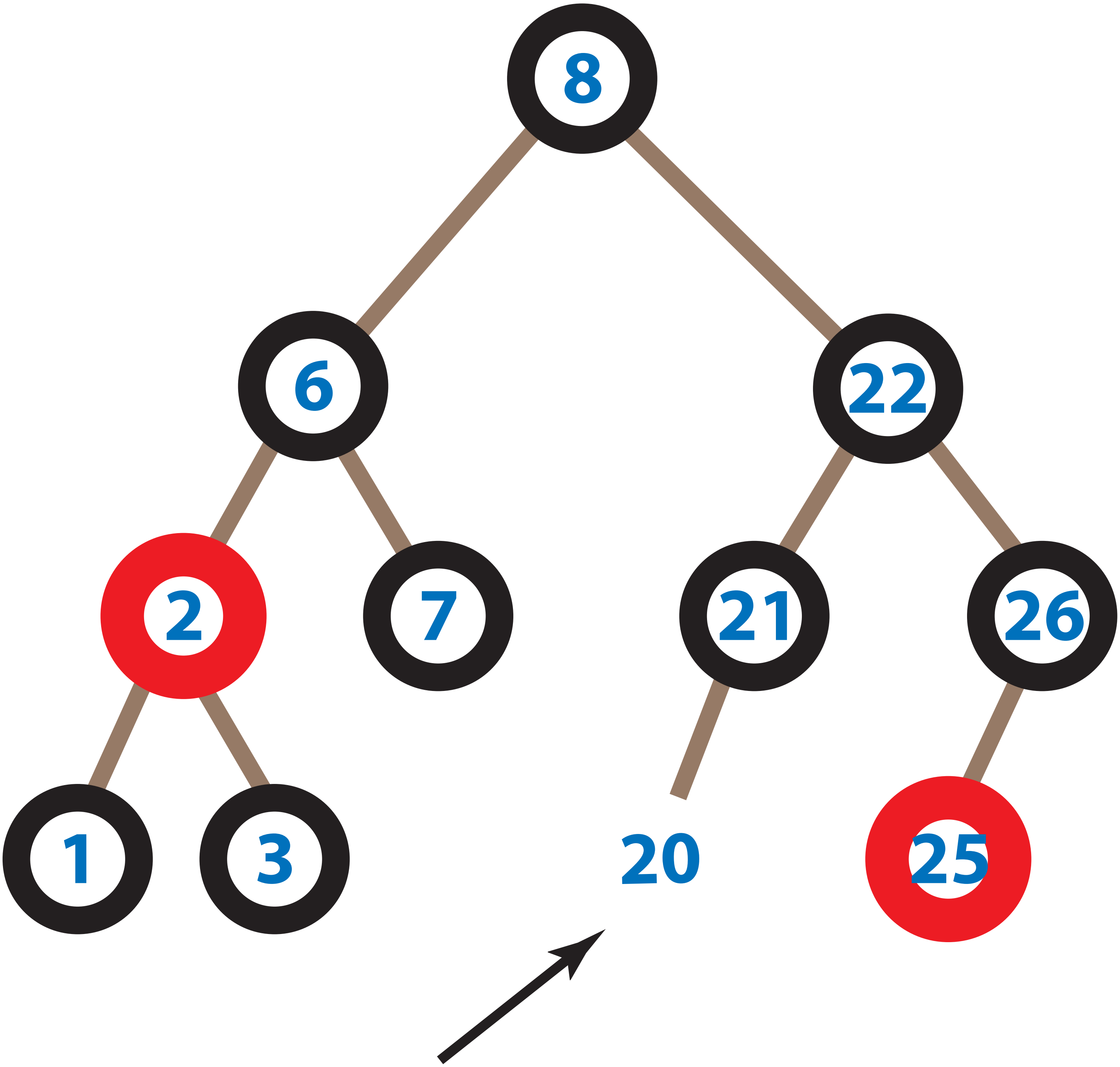


(For presentational simplicity, only showing keys,
and using integer keys not strings.)

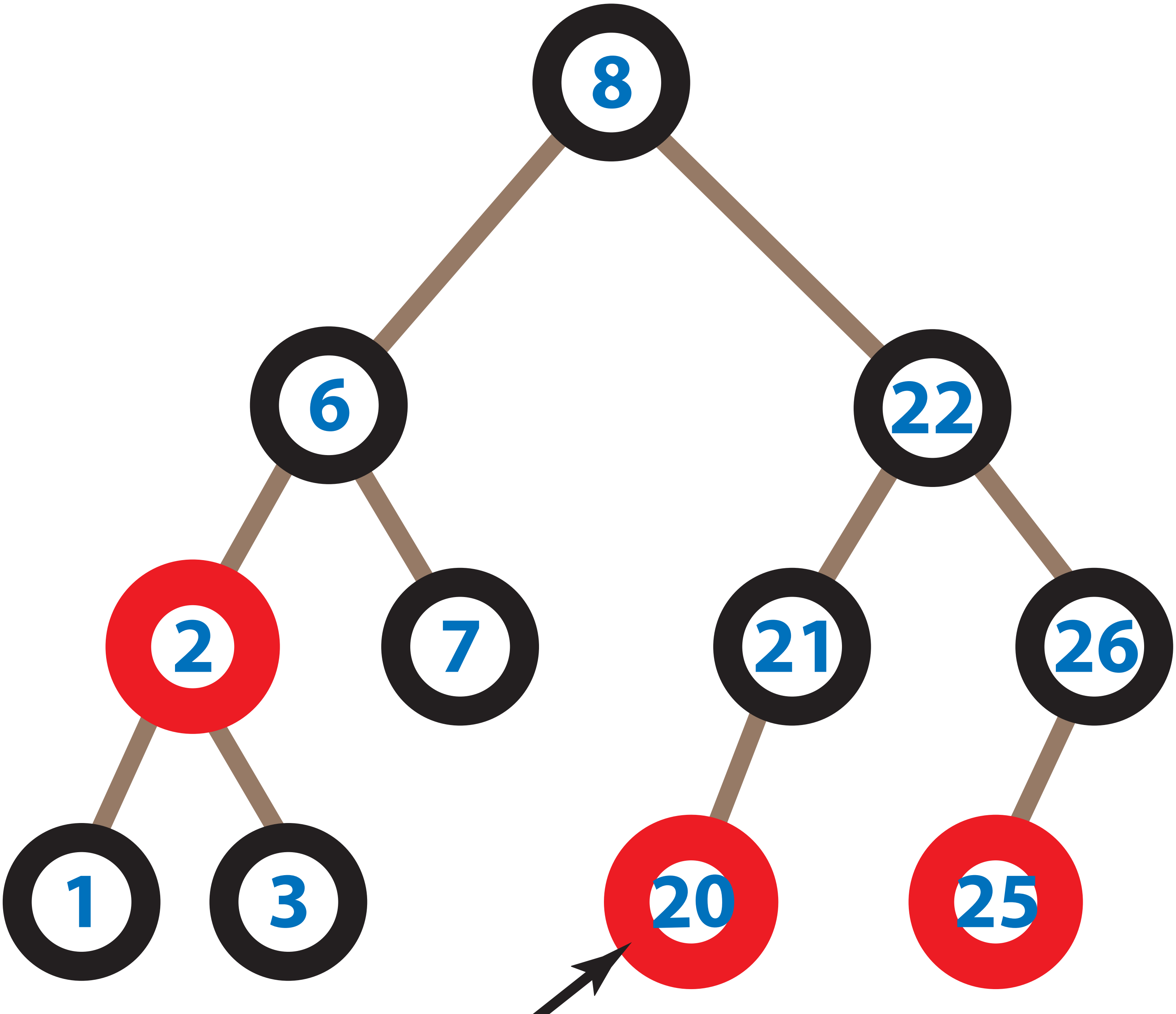
Now insert 20:



Now insert 20:

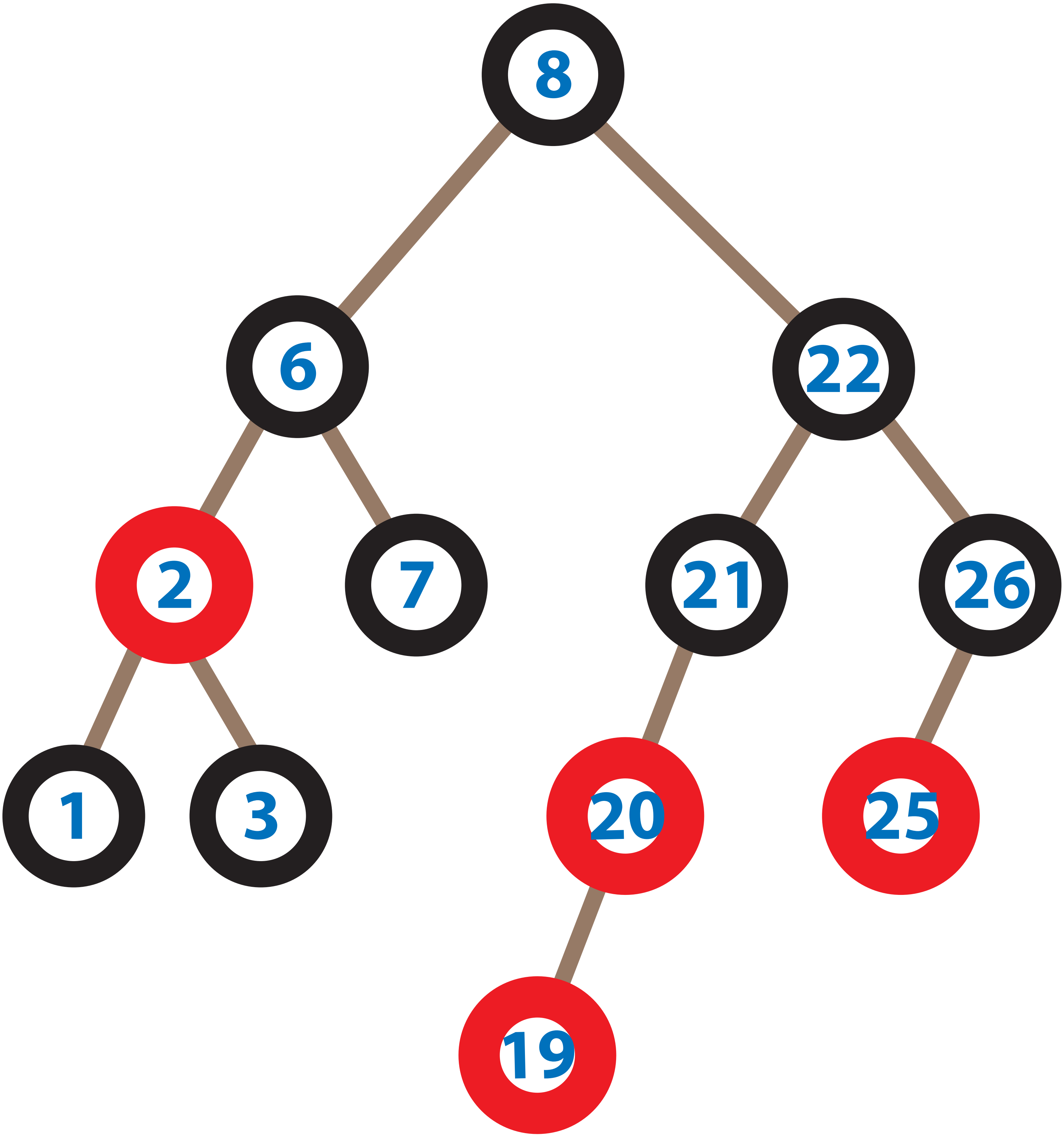


What should we color this node?

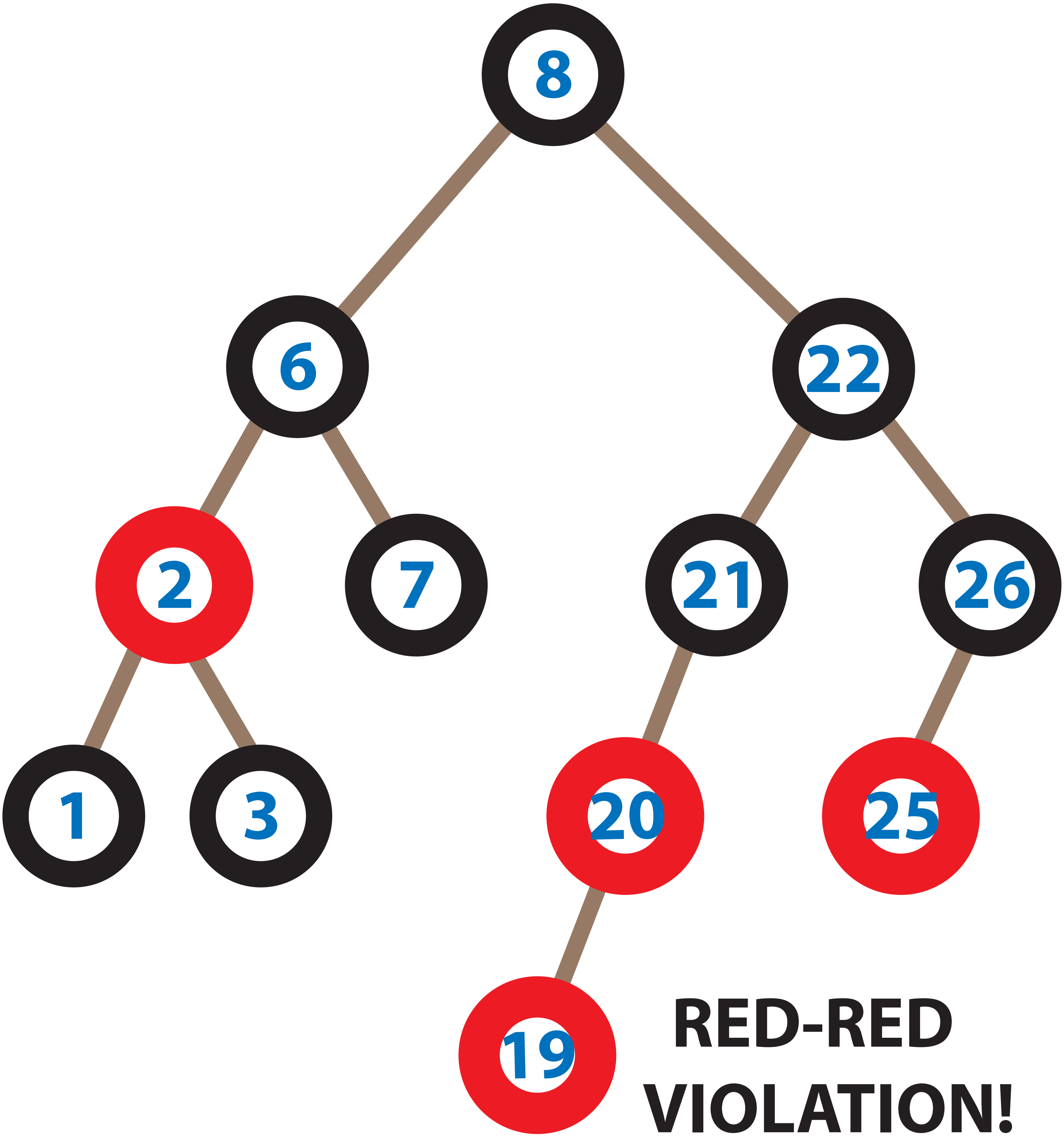


Let’s color it red,
to preserve black height.

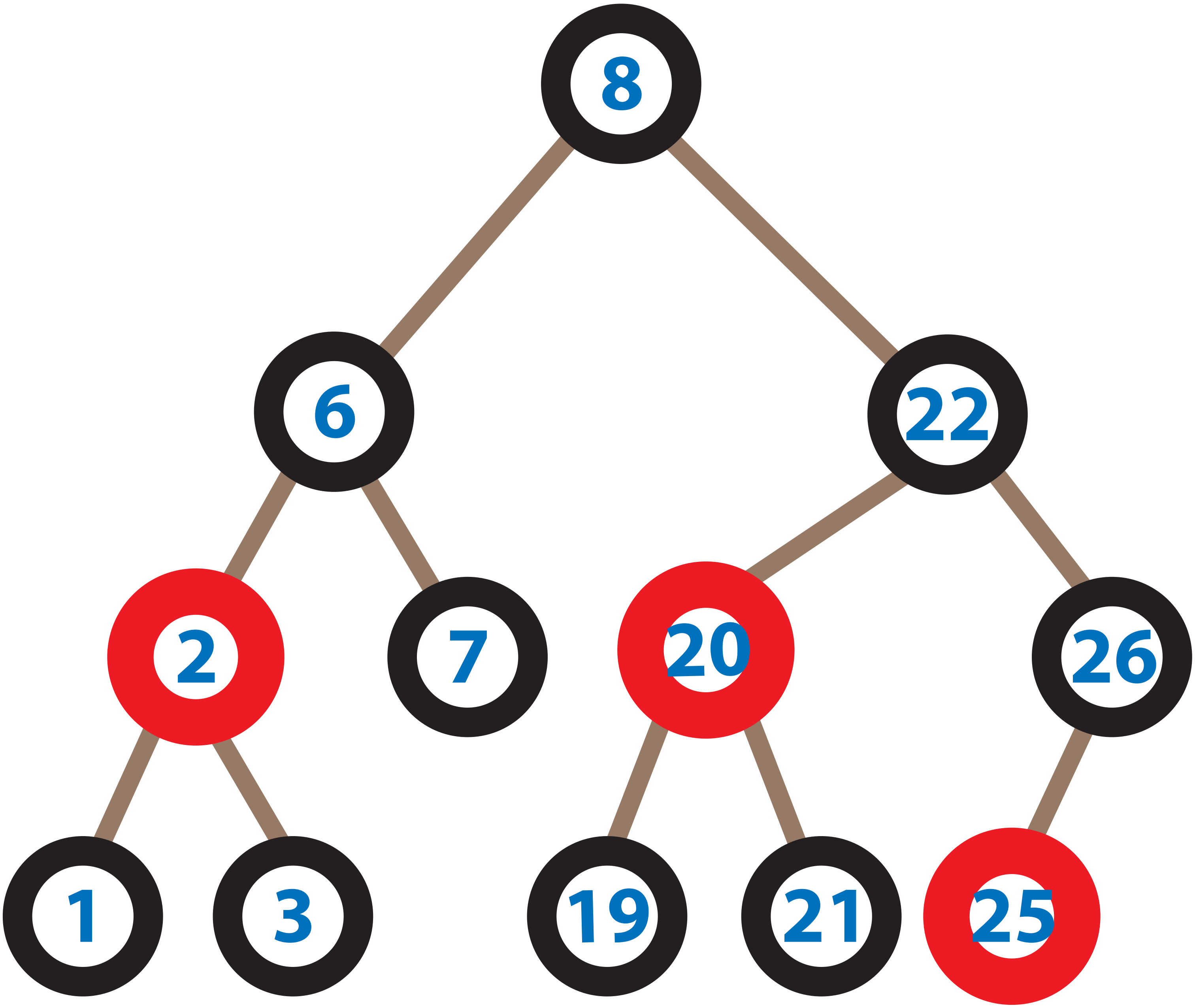
Now insert 19:



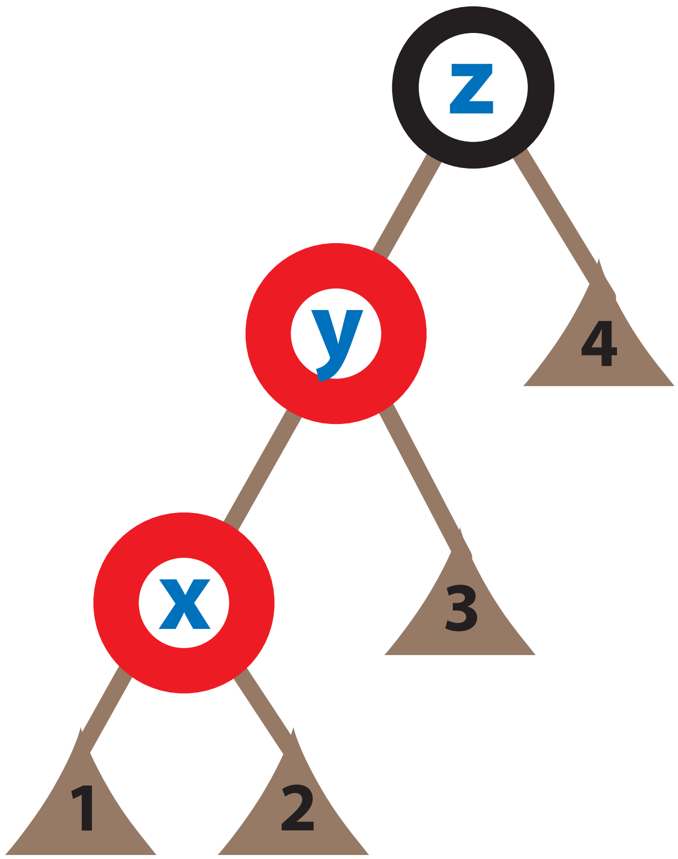
Now insert 19:



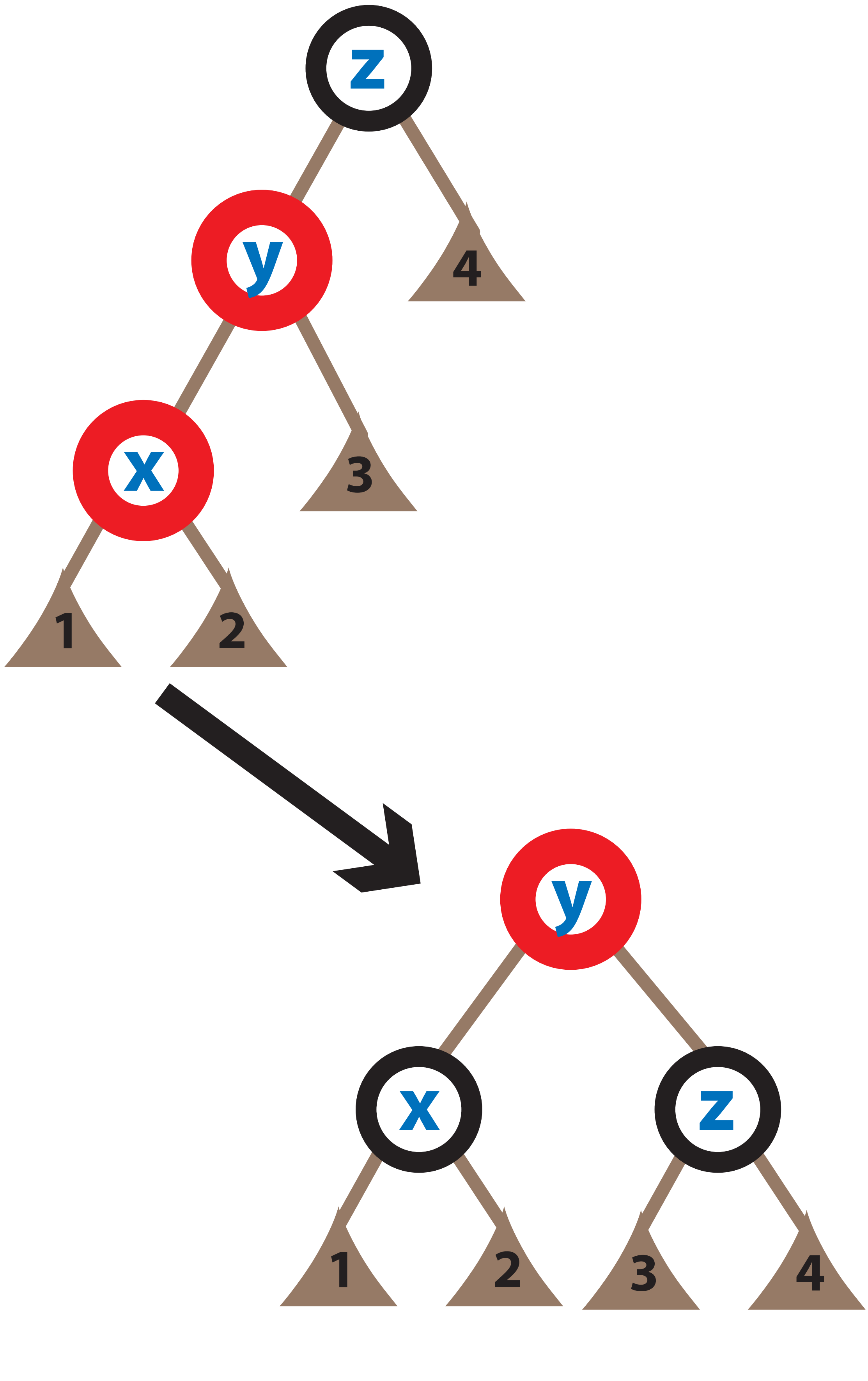
Fix with a rotation and recoloring:



2 of the 4 possible kinds of rotations:



2 of the 4 possible kinds of rotations:



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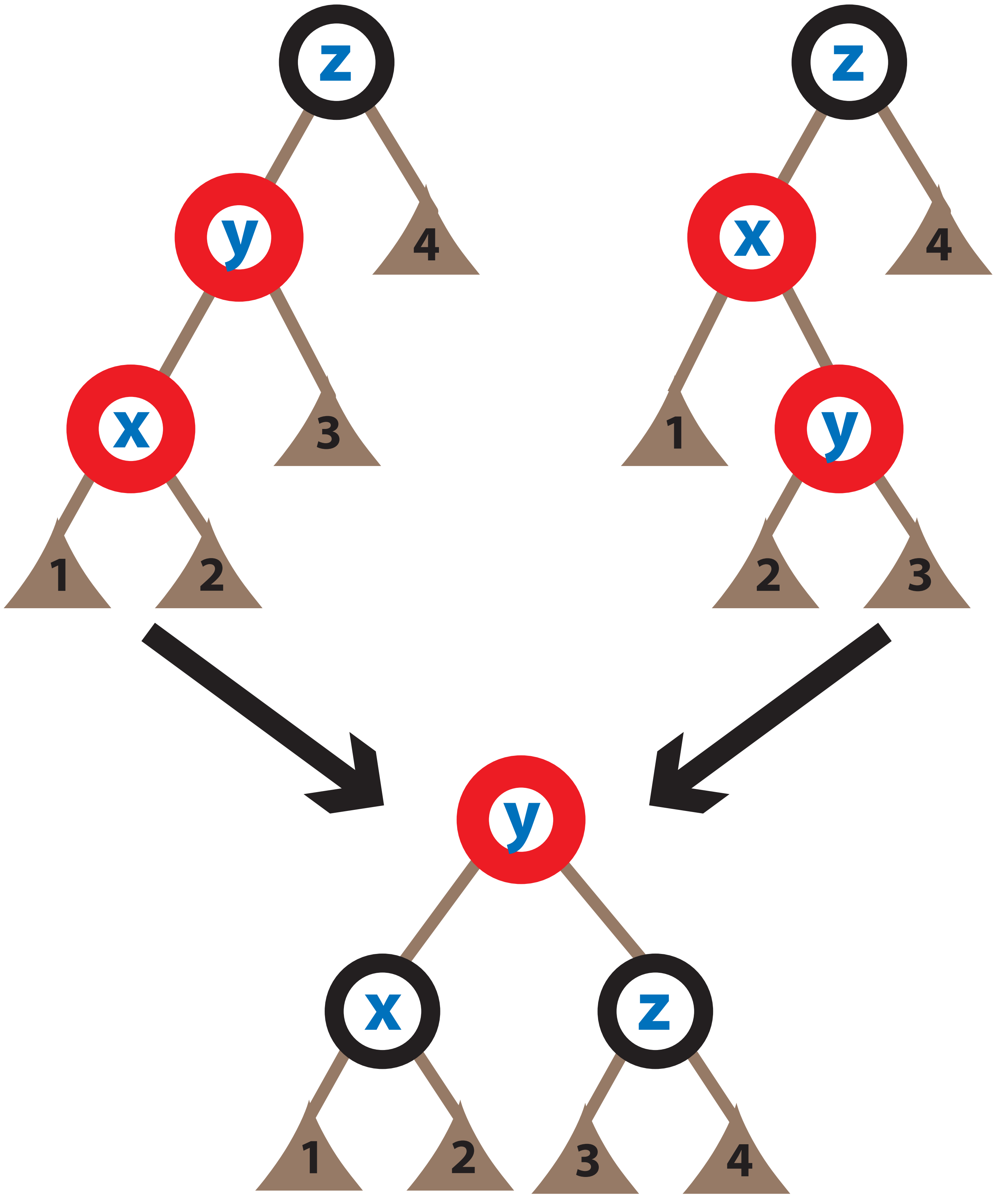

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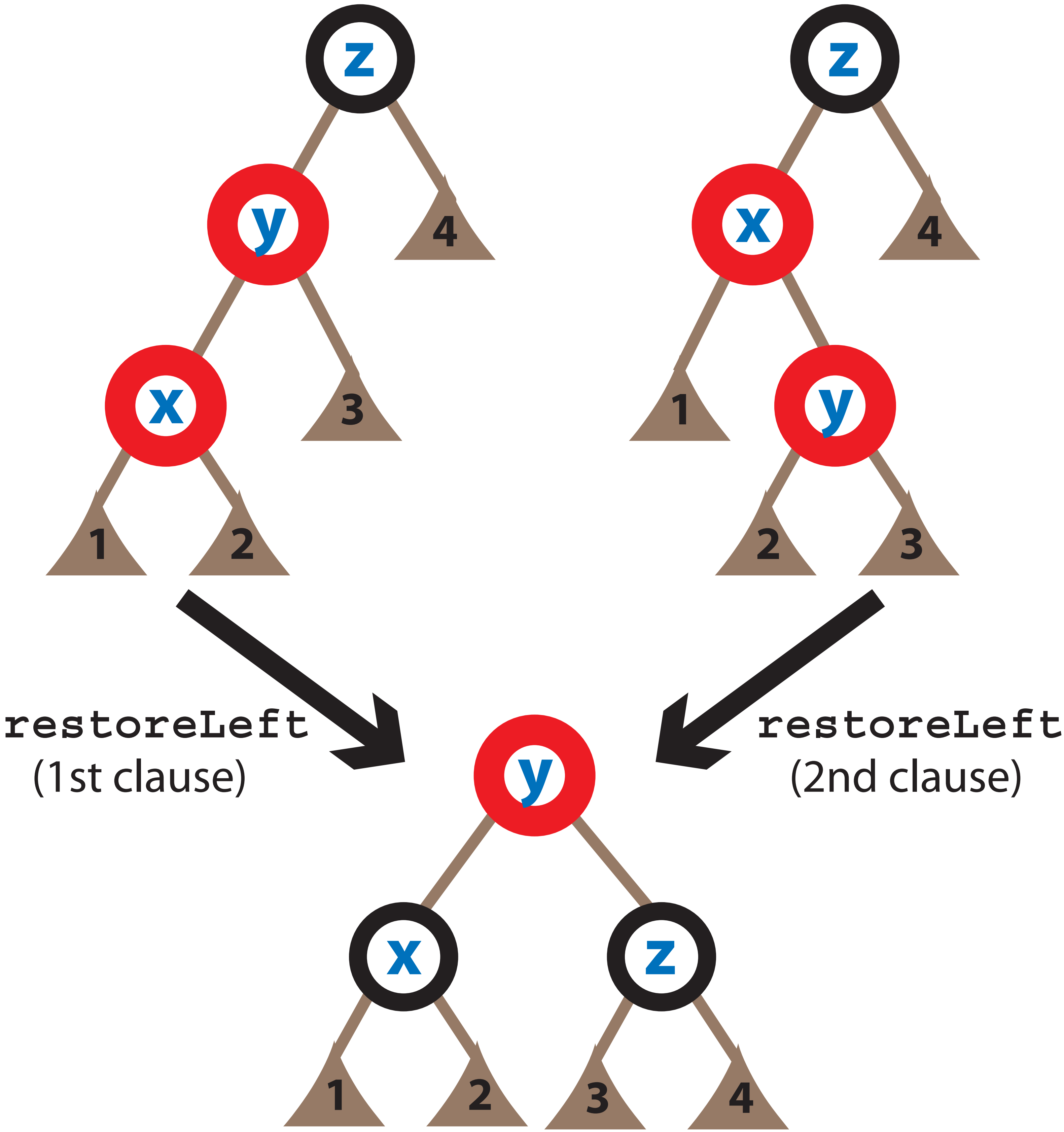
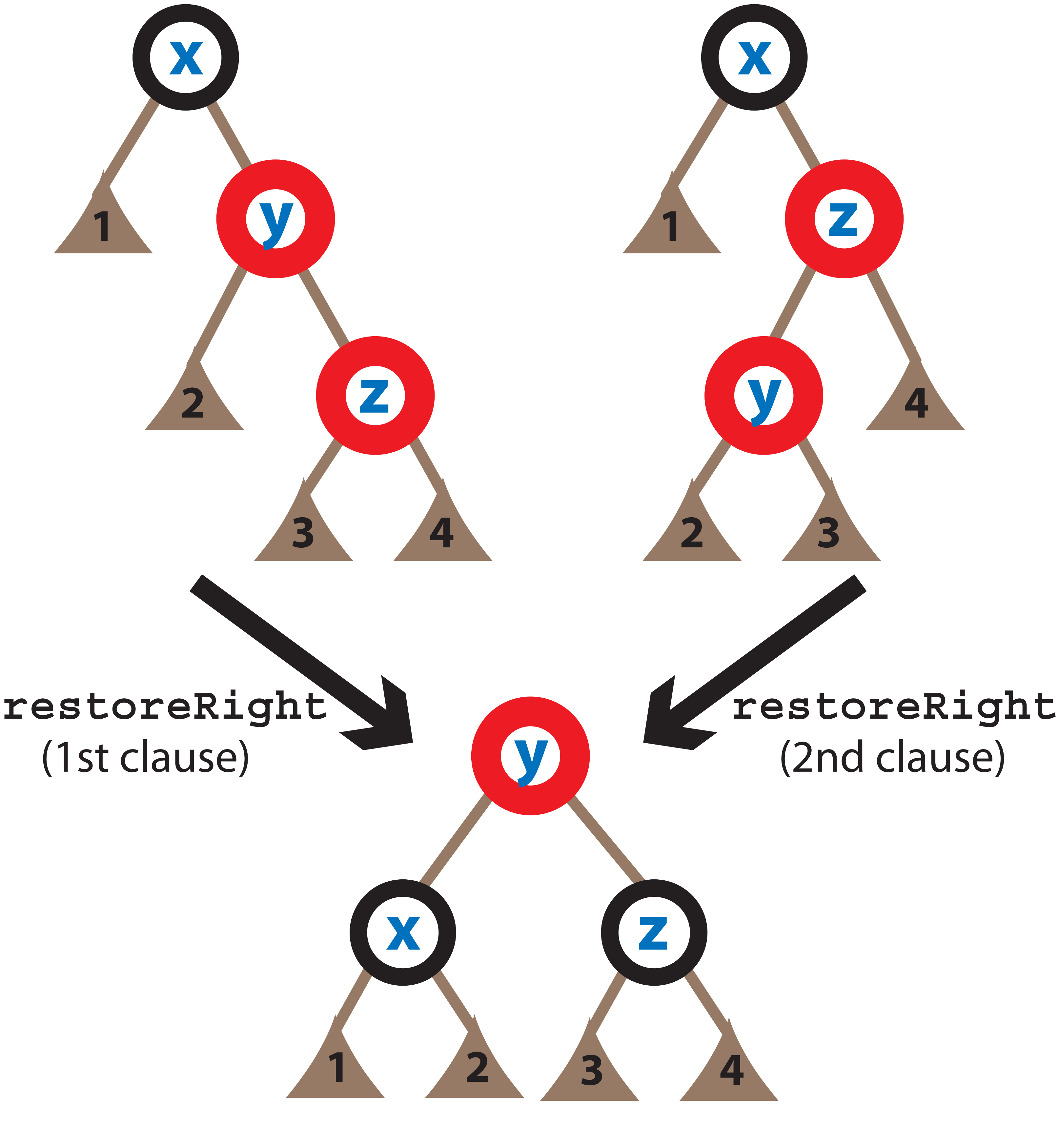
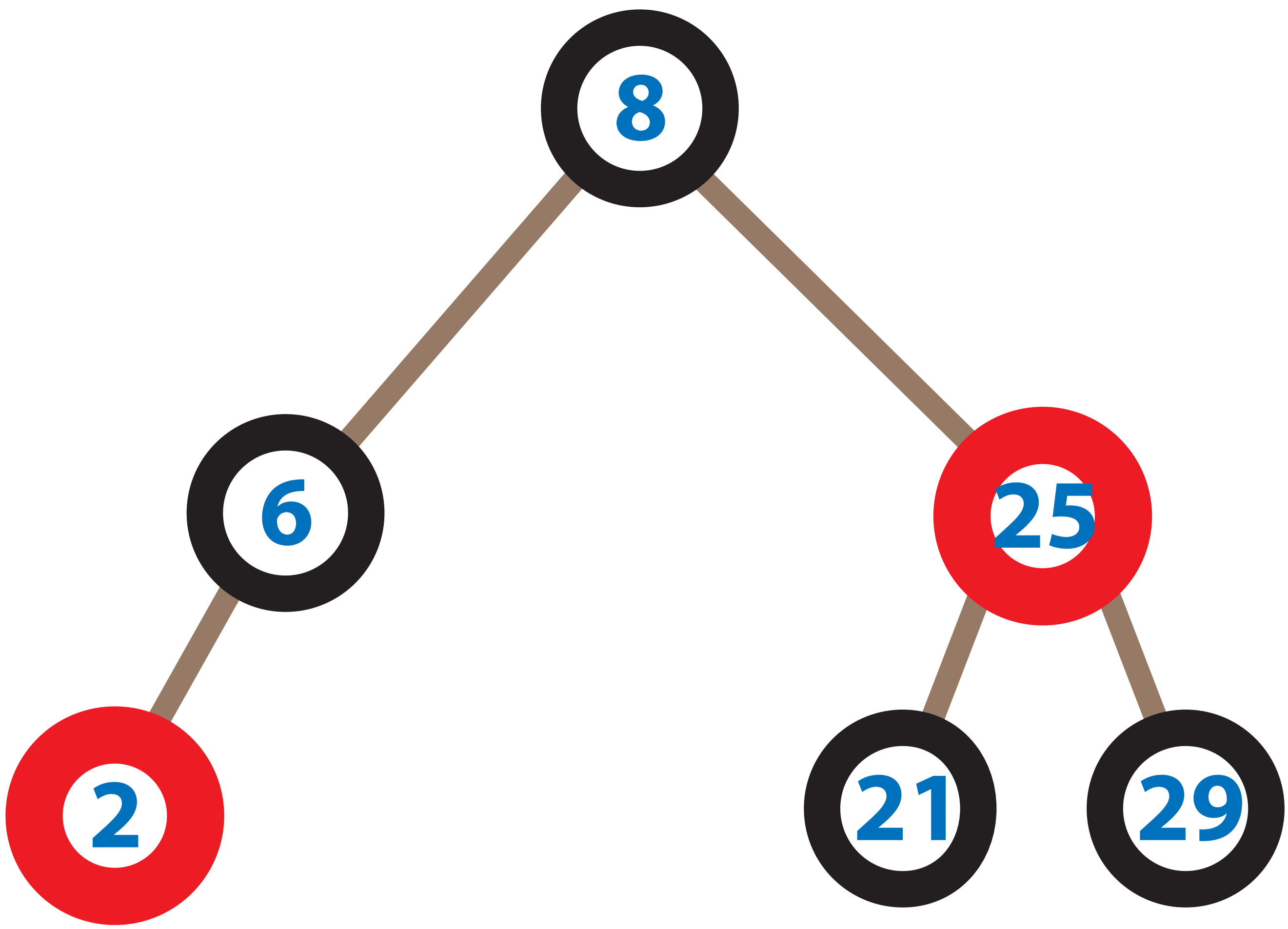


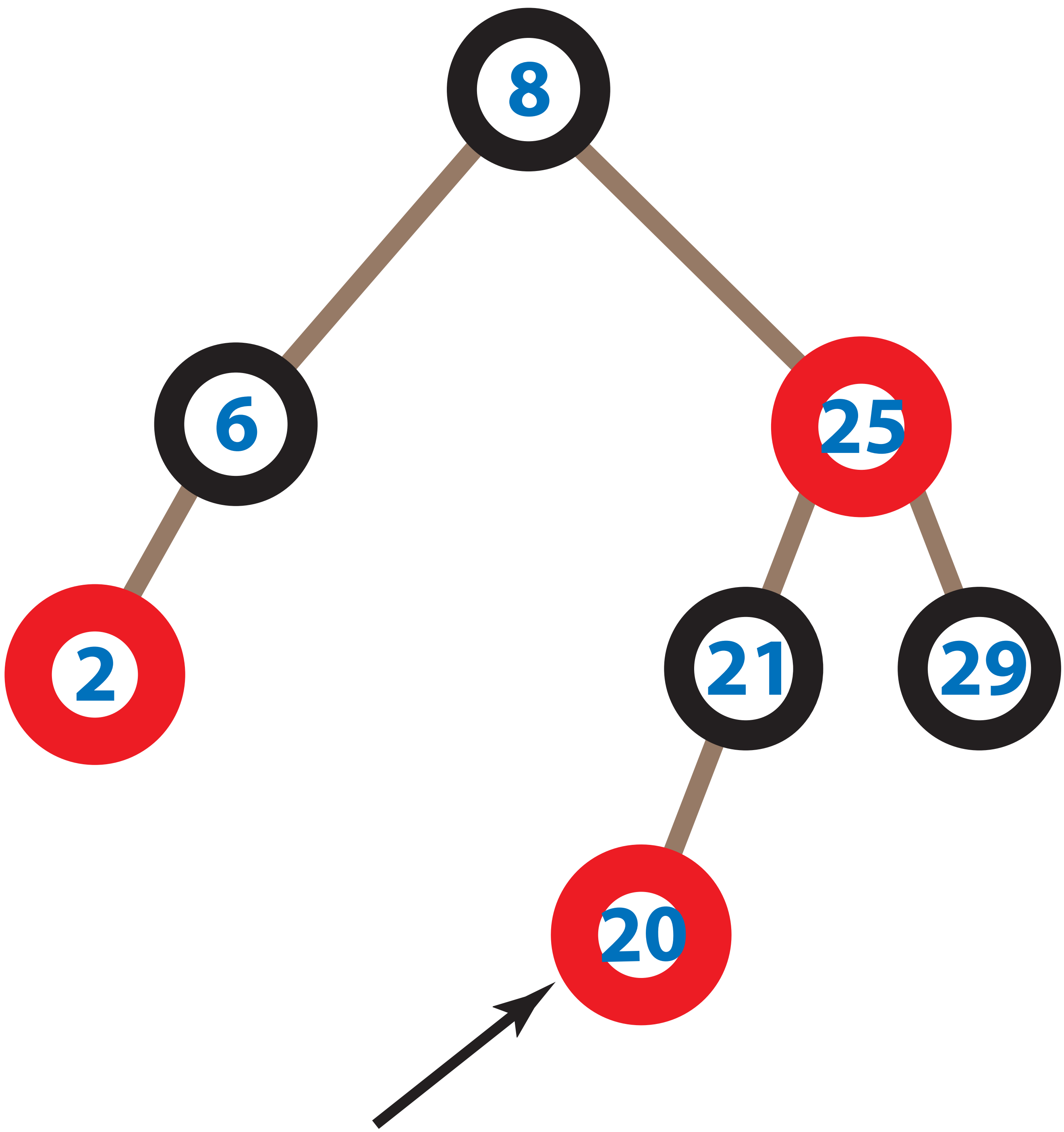
The other 2 kinds of rotations:



Here is another example:

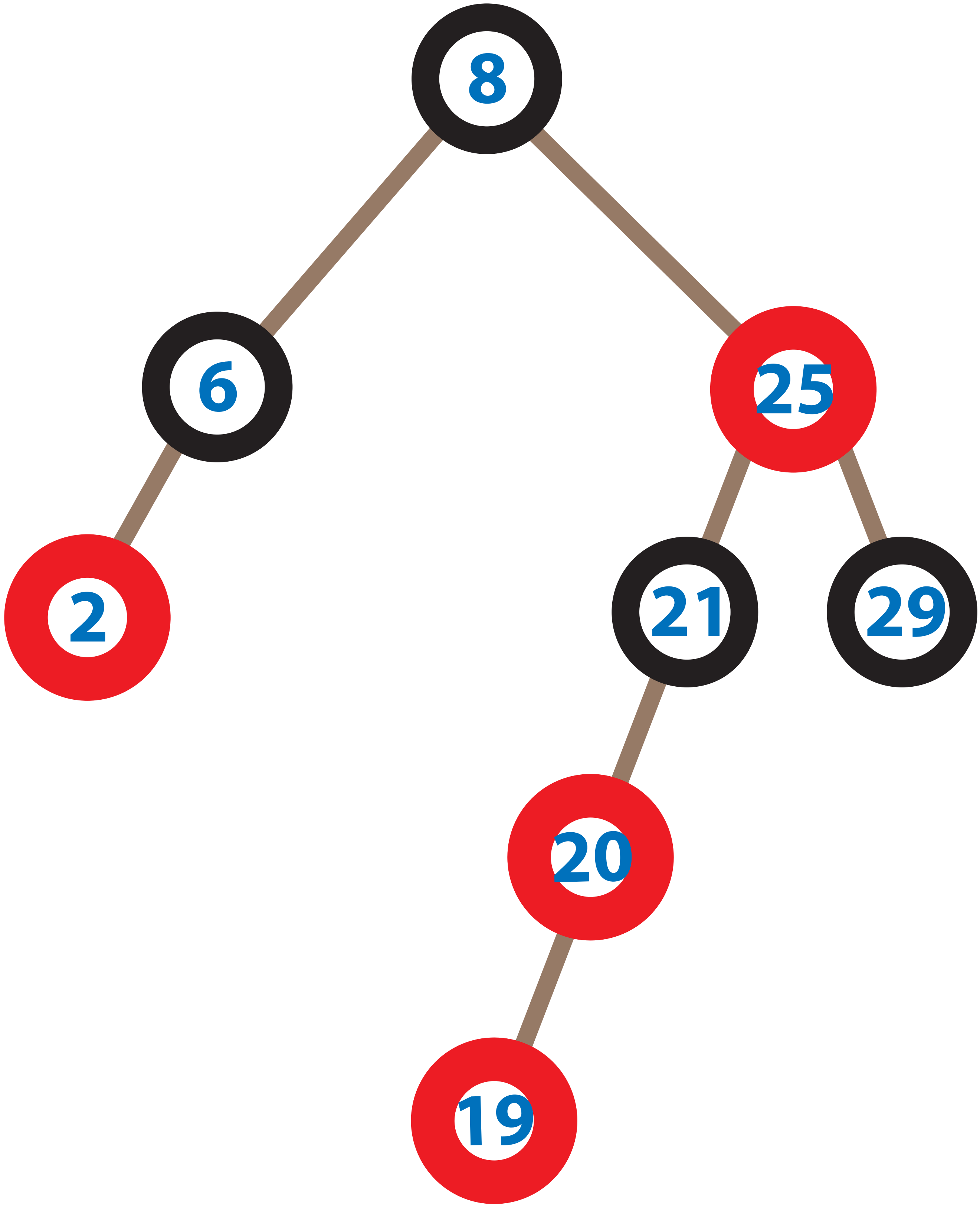


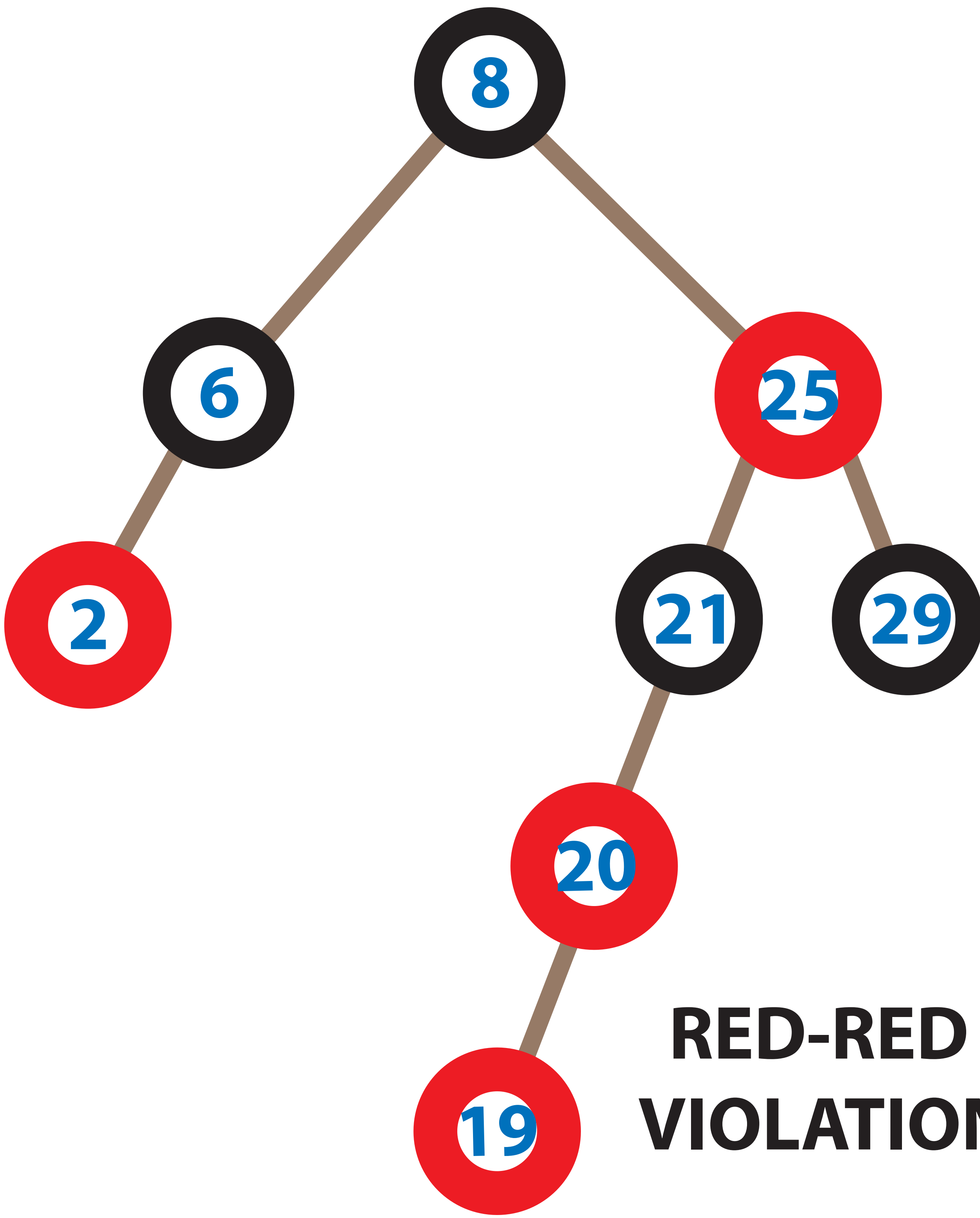
Again, let's insert 20:



Insert 20 and color red (as before)

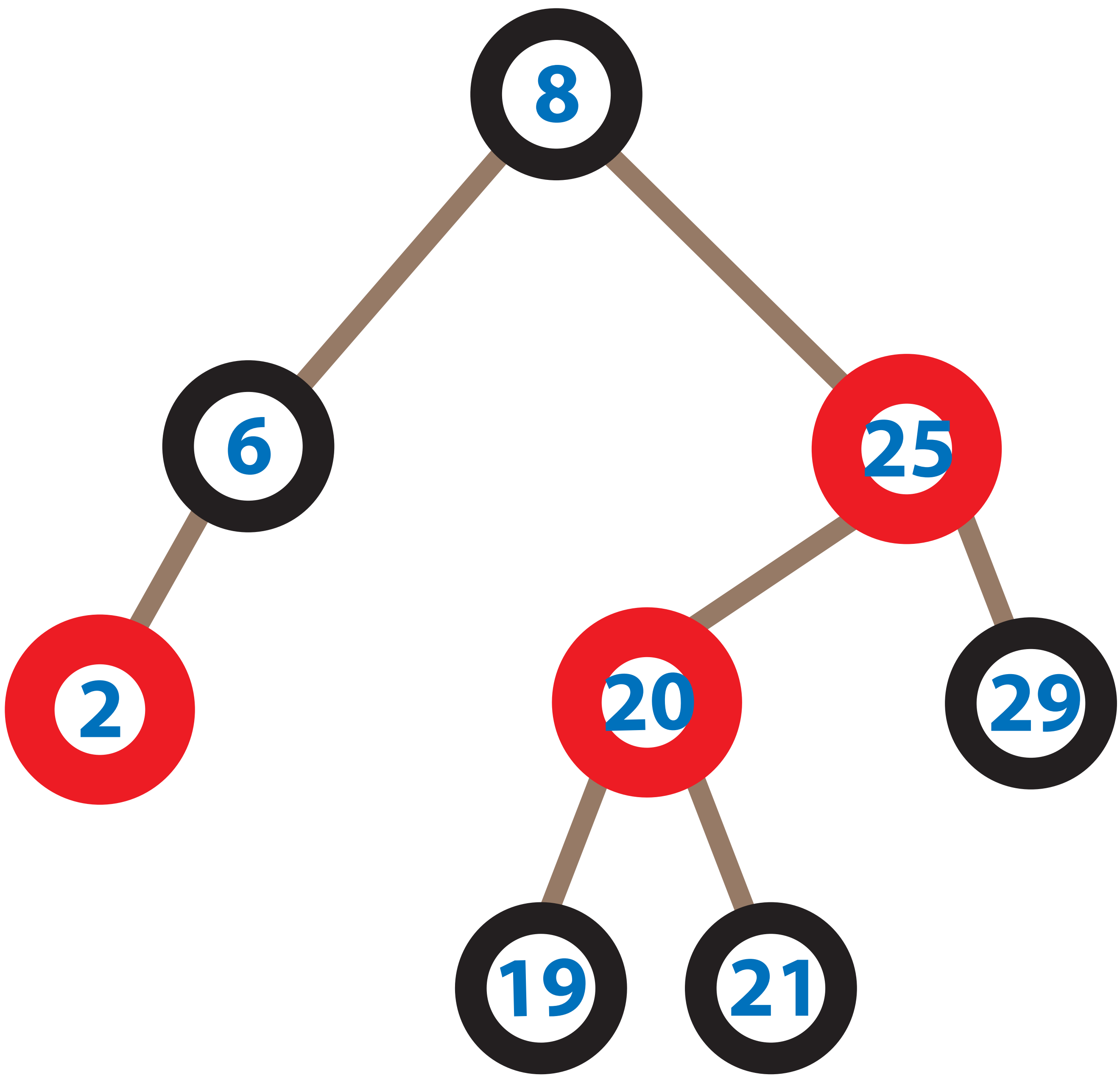
Once again, let's insert 19:

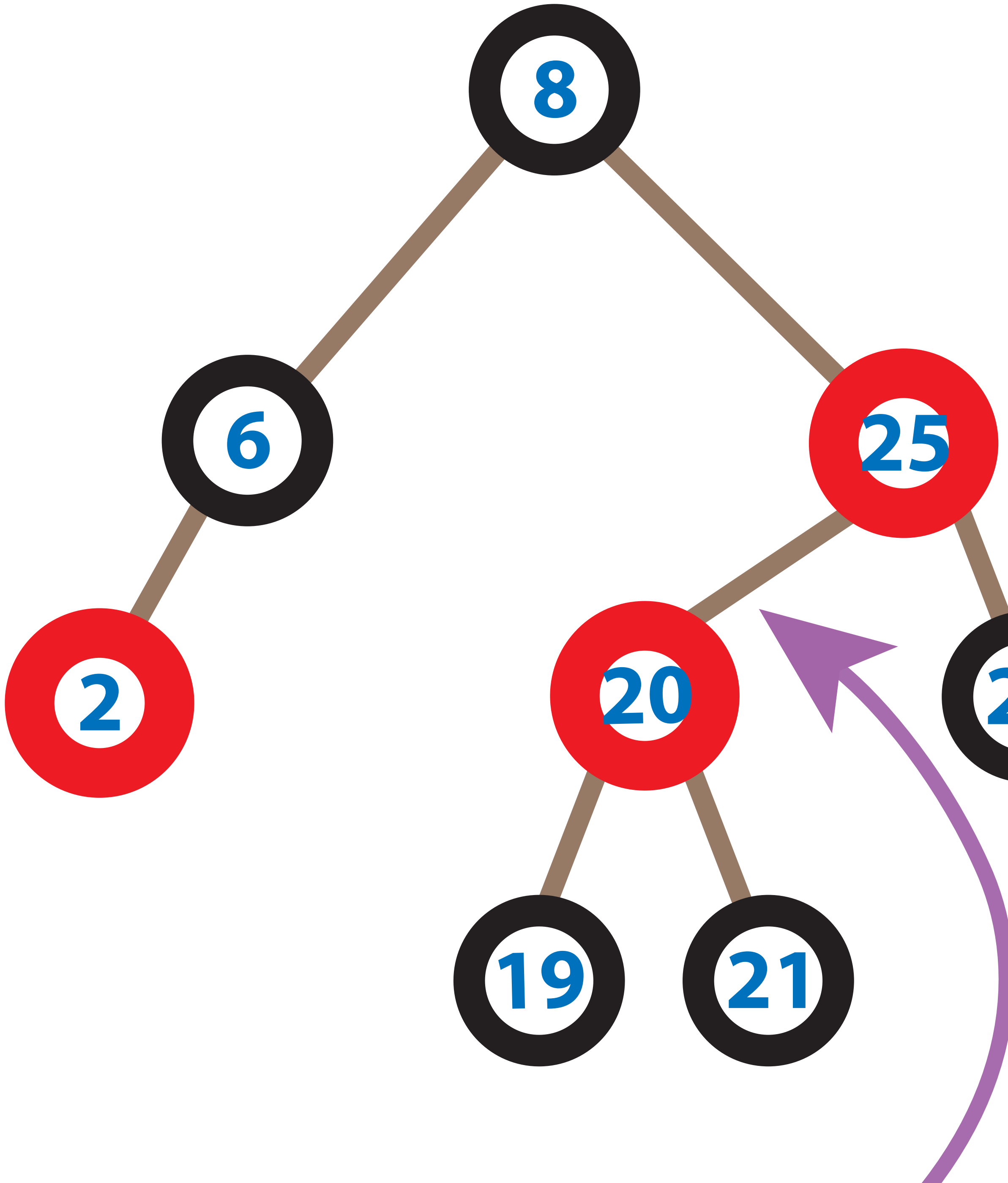




RED-RED
VIOLATION!

Again, fix with rotation & recoloring:

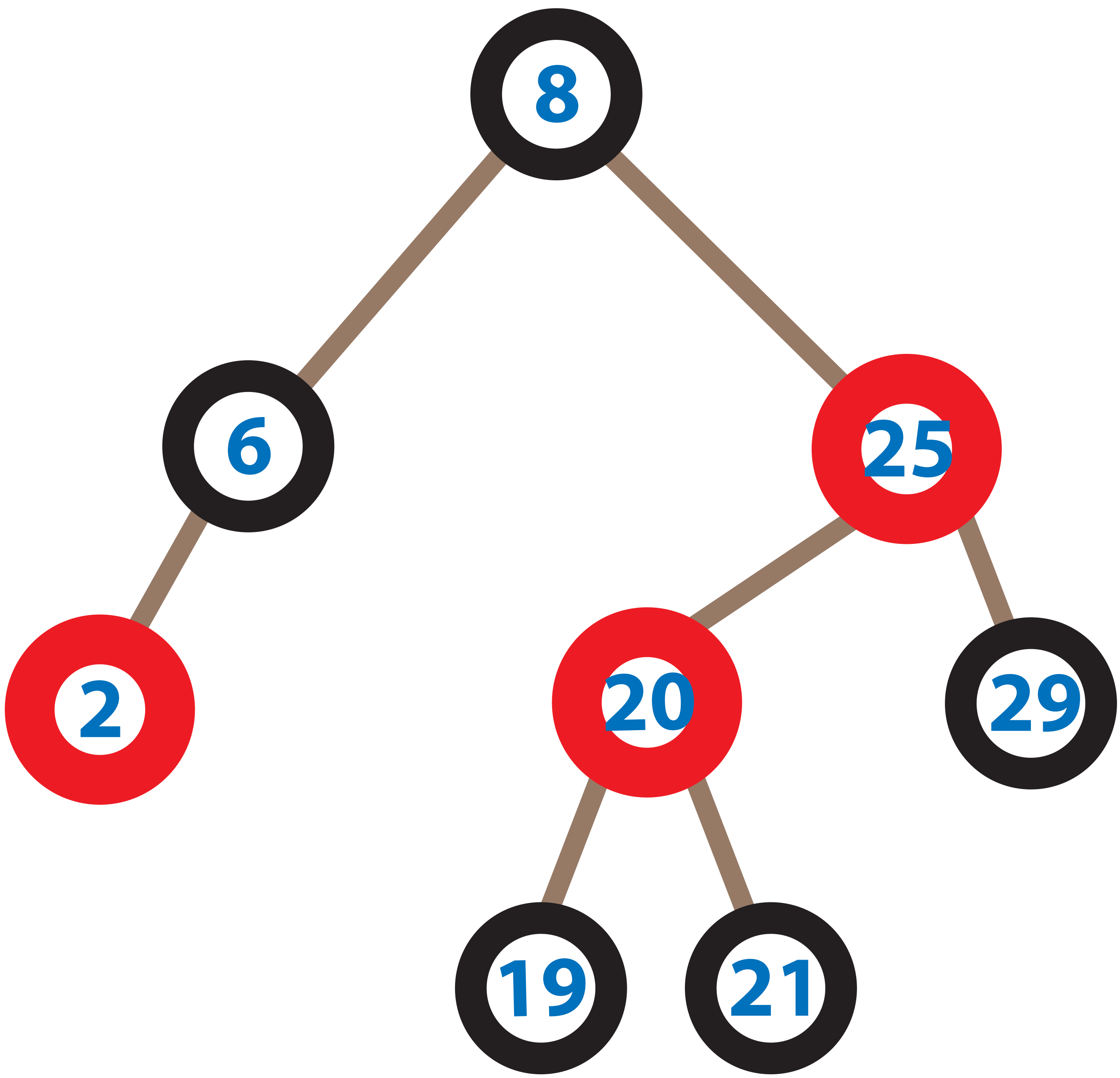




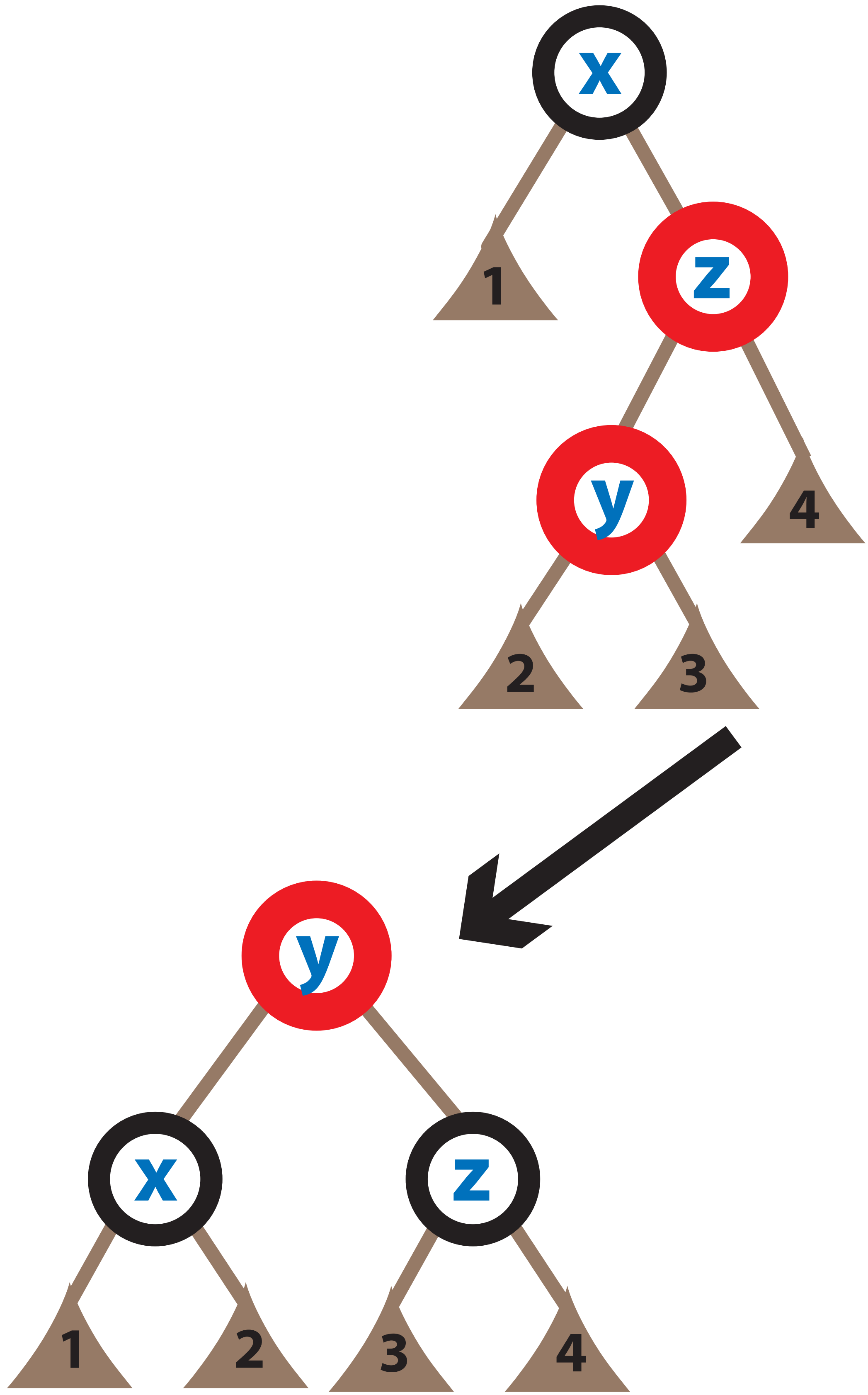
OH NO! There is a new

RED-RED VIOLATION!

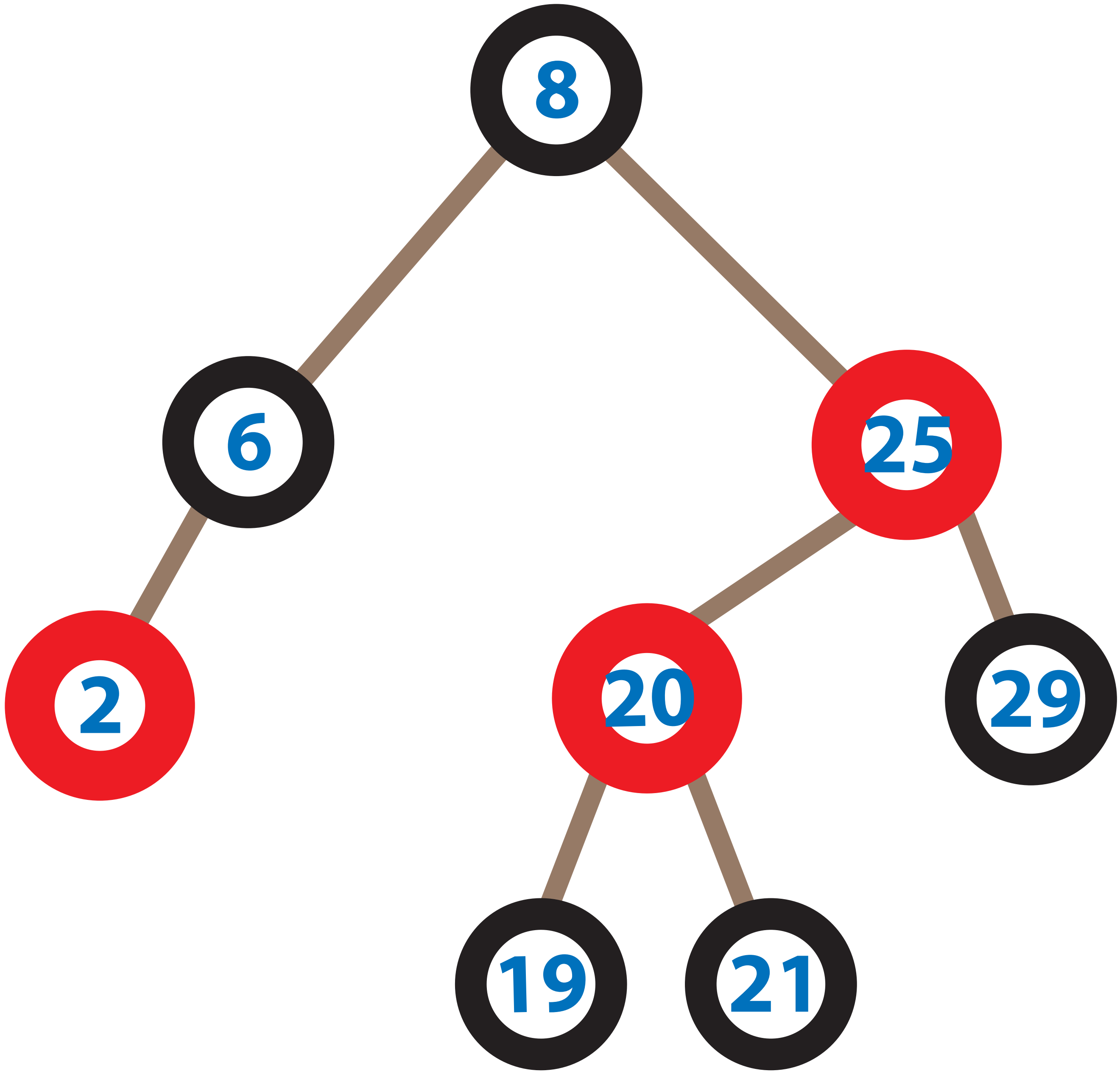
That's OK. We can rotate again ...



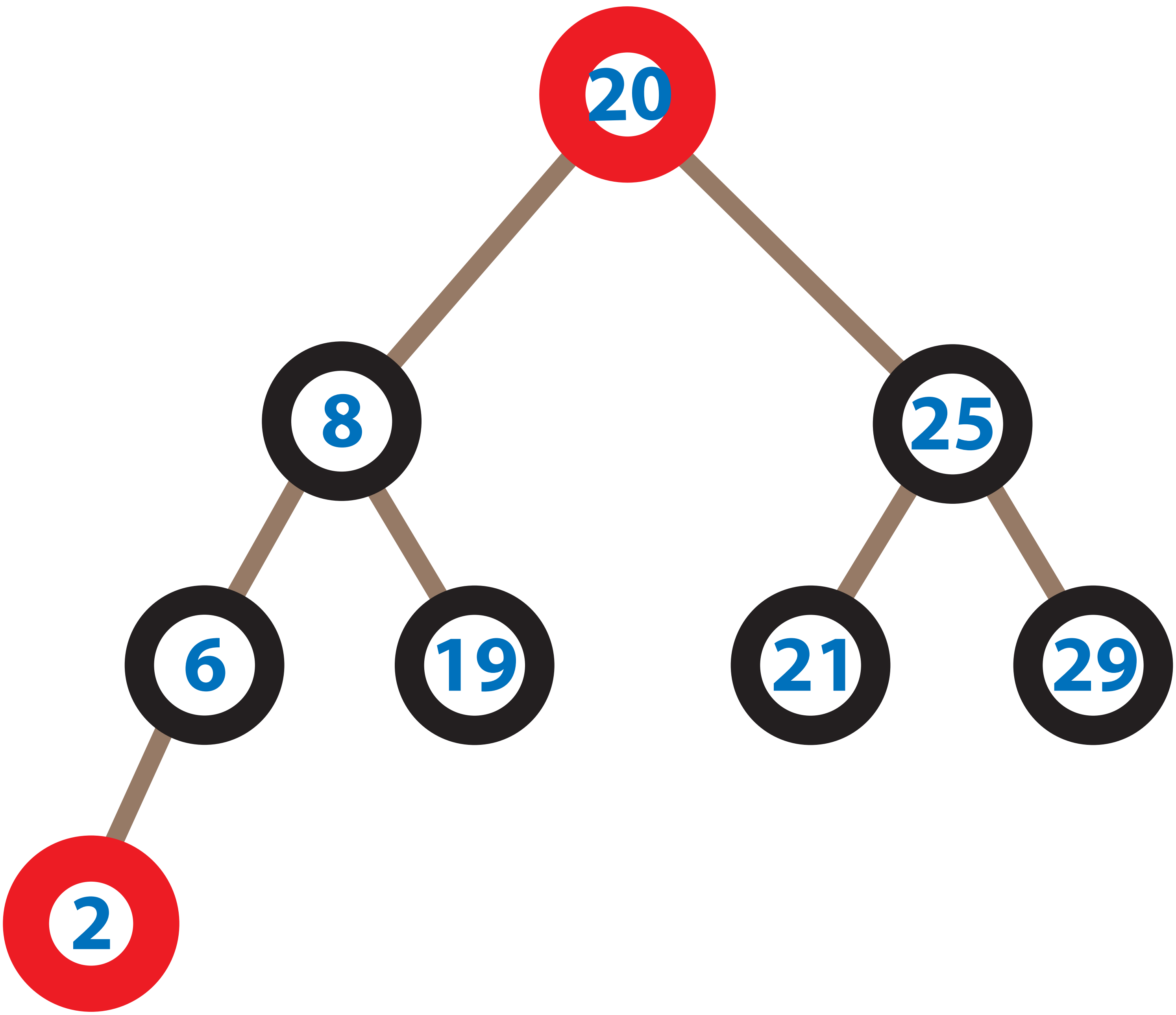
Use this kind of rotation:

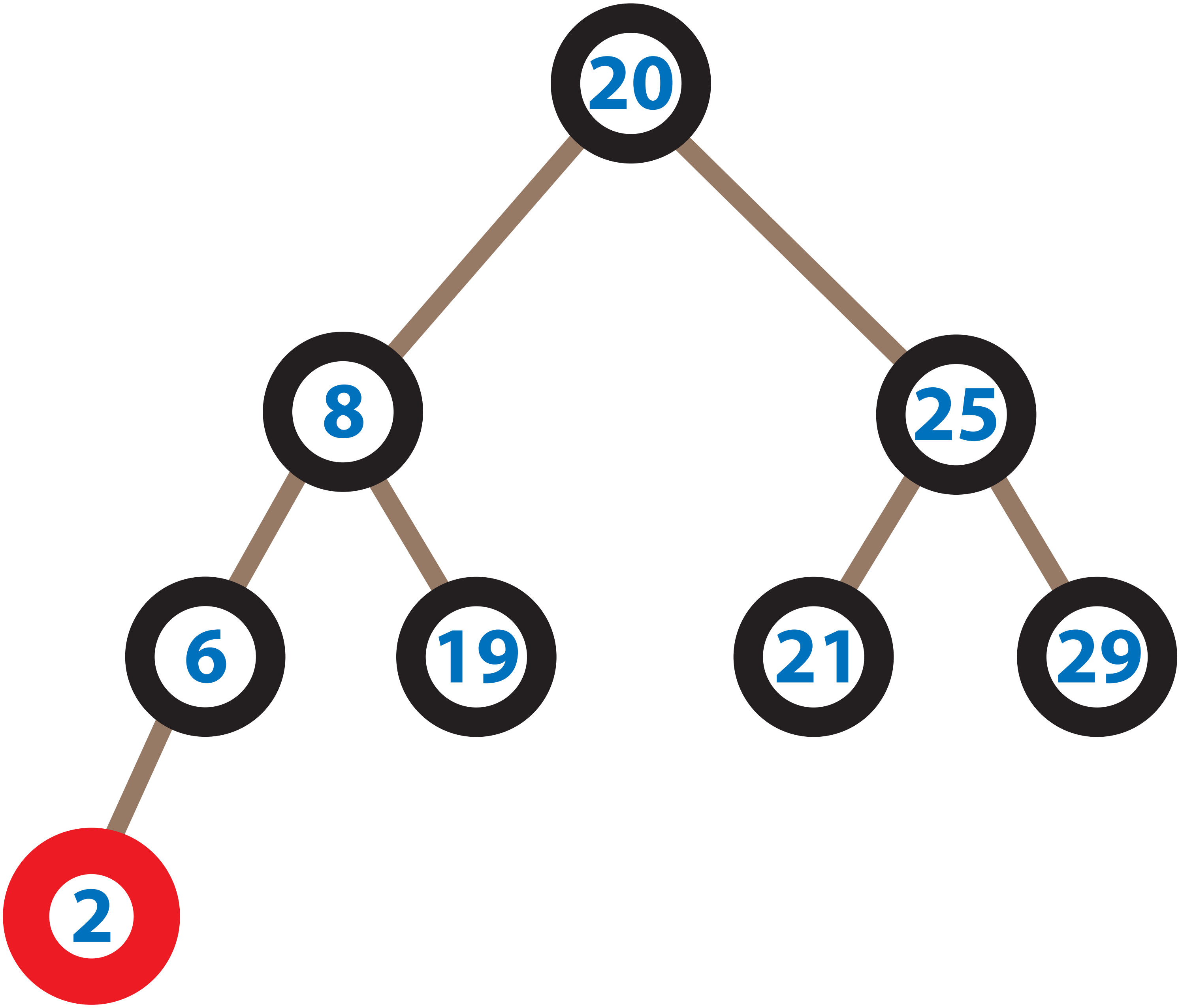


Here's the tree again before rotation:



... giving us this after the rotation:





(It's not necessary, but we can also safely recolor the root black.)

Red Black Tree Dictionaries

Binary search tree with Red and Black nodes:

```
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   Empty
 | Red of 'a dict * 'a entry * 'a dict
 | Black of 'a dict * 'a entry * 'a dict
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The number of `Black` nodes on *any* path
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Almost RBT (ARBT) Invariants:

(1) and (3) as above.

(2′) Like (2), but: `Red` **root** may have **one** `Red` child.

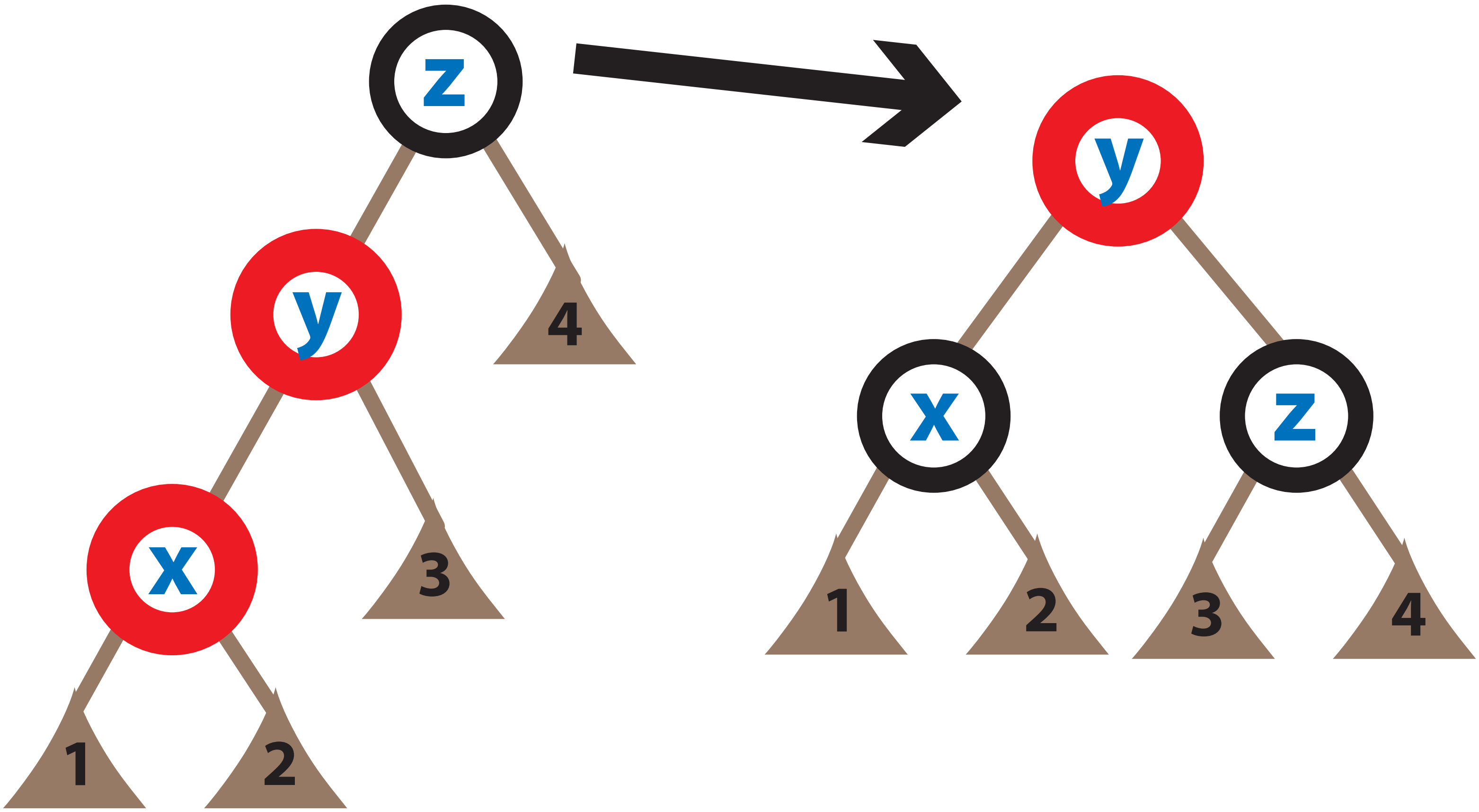
Specs for `restoreLeft`

```
(*
  restoreLeft : 'a dict -> 'a dict

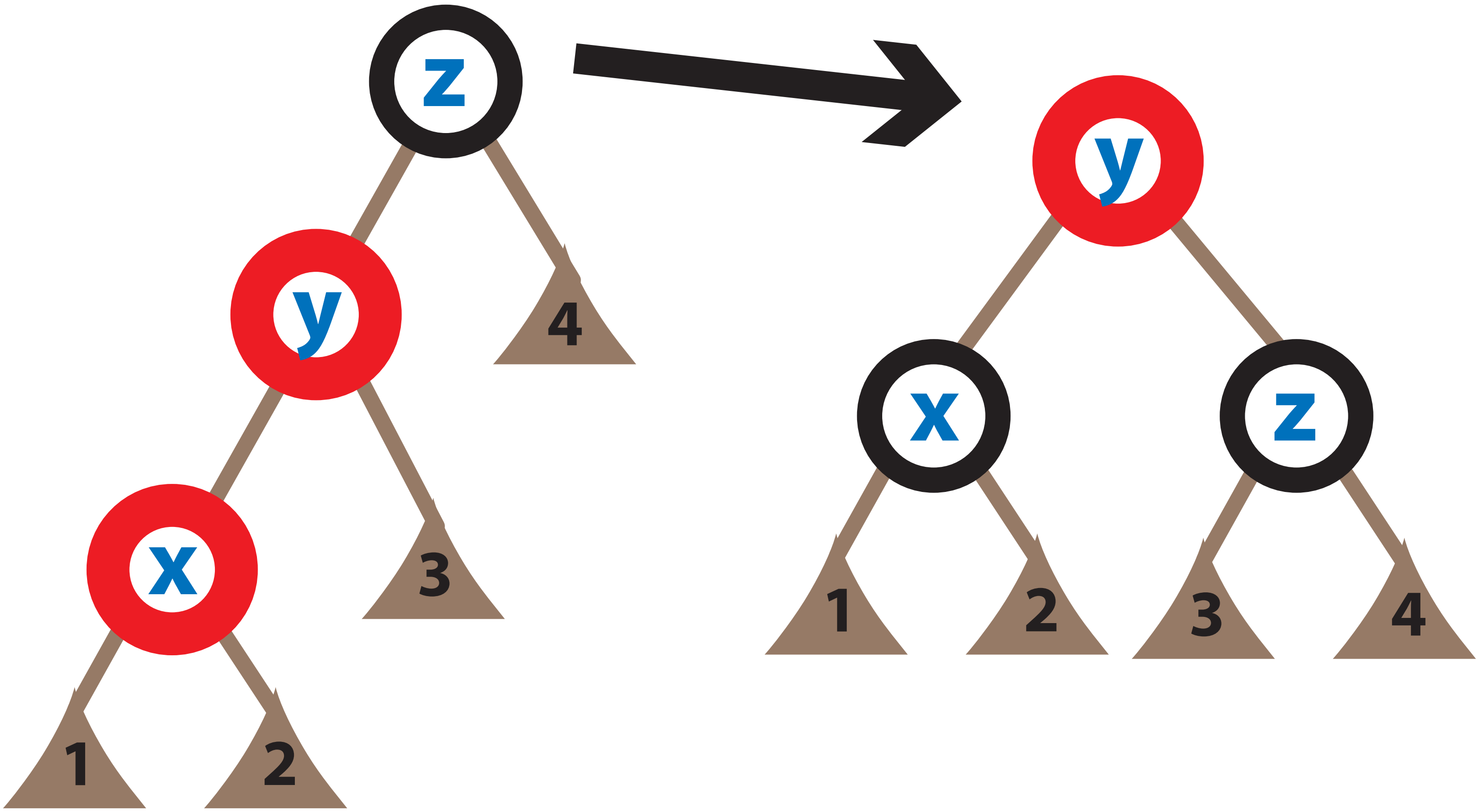
  REQUIRES: Either d is a RBT
            or d's root is Black,
               its left child is an ARBT,
               and its right child a RBT.

  ENSURES: restoreLeft(d) is a RBT,
           containing exactly the same
           entries as d, and with the
           same black height as d.
*)
```

Picture-based Programming

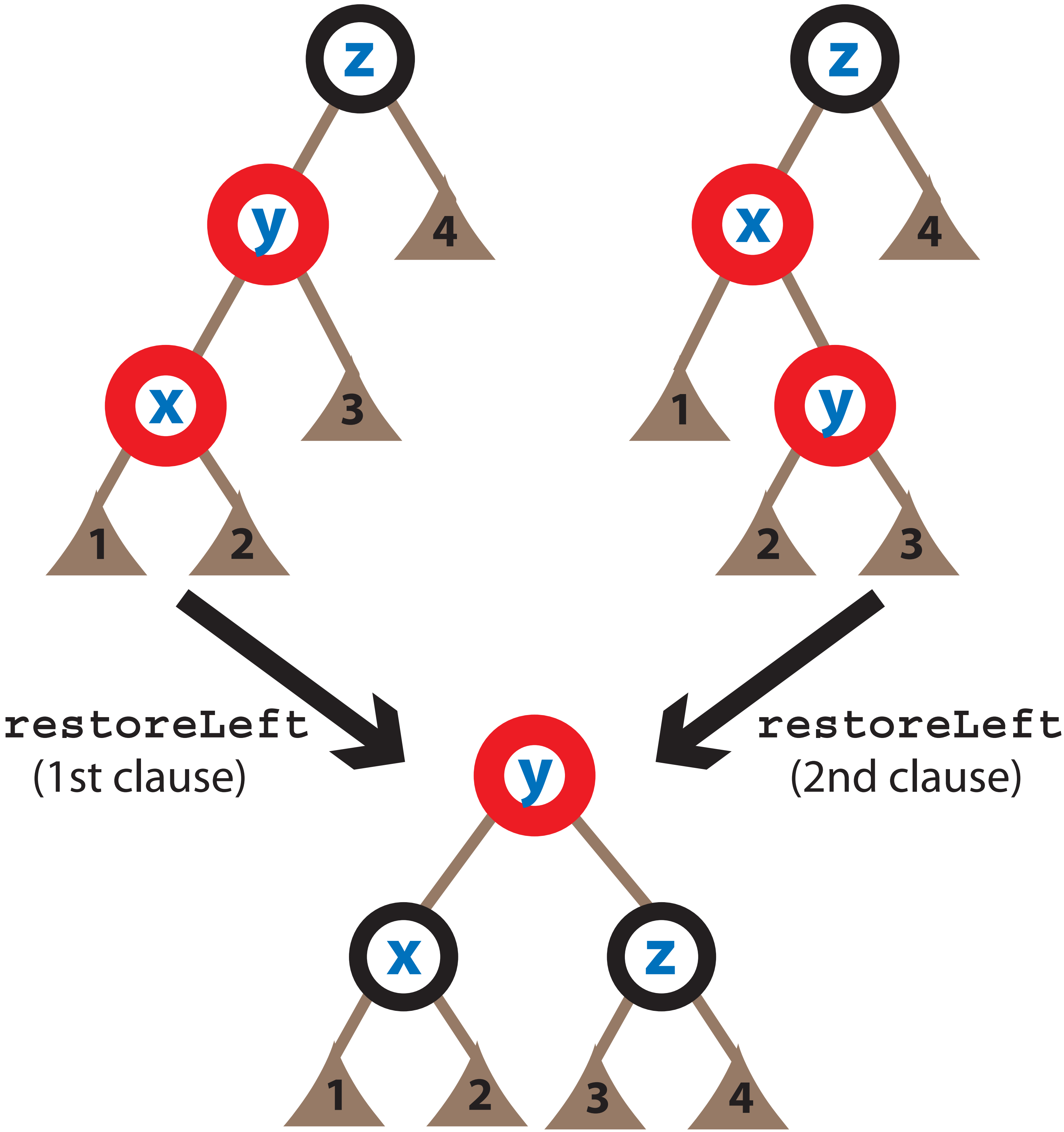


Picture-based Programming




```
fun
 restoreLeft
    (Black(Red(Red(d1, x, d2), y, d3), z, d4)) =
       Red(Black(d1, x, d2), y, Black(d3, z, d4))
```

2 of the 4 possible kinds of rotations:



Code for `restoreLeft`

```
(*
  restoreLeft : 'a dict -> 'a dict

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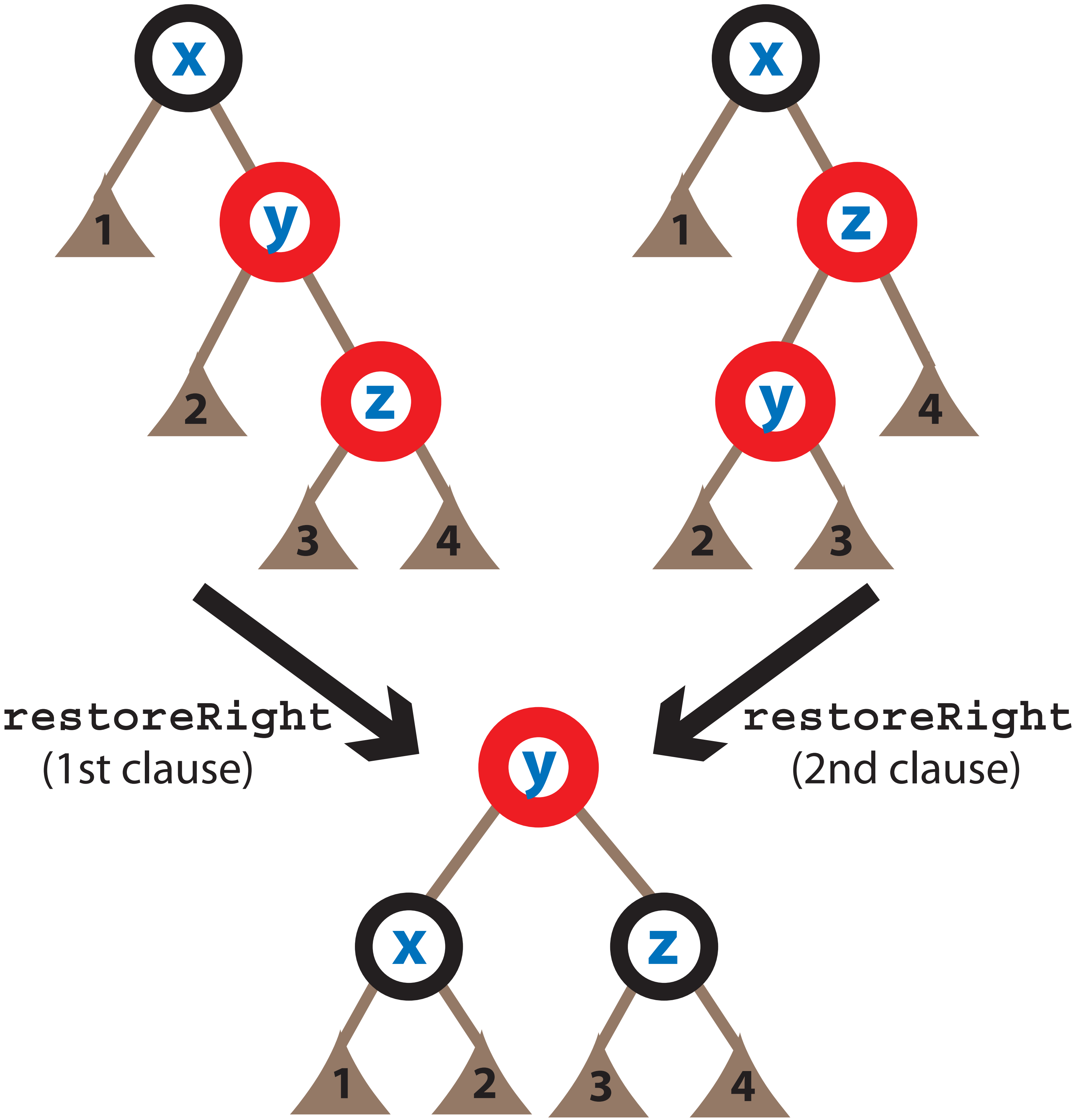
  ENSURES: restoreLeft(d) is a RBT,
           containing exactly the same
           entries as d, and with the
           same black height as d.
*)

fun
 restoreLeft(Black(Red(Red(d1,x,d2),y,d3),z,d4))=
     Red(Black(d1,x,d2), y, Black(d3,z,d4))

|restoreLeft(Black(Red(d1,x,Red(d2,y,d3)),z,d4))=
     Red(Black(d1,x,d2), y, Black(d3,z,d4))

|restoreLeft d = d
```

The other 2 kinds of rotations:



Code for `restoreRight`

```
(*
  restoreRight : 'a dict -> 'a dict

  REQUIRES: Either d is a RBT
            or d's root is Black,
               its right child is an ARBT,
               and its left child a RBT.

  ENSURES: restoreRight(d) is a RBT,
           containing exactly the same
           entries as d, and with the
           same black height as d.
*)

fun
 restoreRight(Black(d1,x,Red(d2,y,Red(d3,z,d4))))=
     Red(Black(d1,x,d2), y, Black(d3,z,d4))

|restoreRight(Black(d1,x,Red(Red(d2,y,d3),z,d4)))=
     Red(Black(d1,x,d2), y, Black(d3,z,d4))

|restoreRight d = d
```

Dictionary Signature

```
signature DICT =
sig
  type key = string              (* concrete *)
  type 'a entry = key * 'a       (* concrete *)

  type 'a dict                   (* abstract *)

  val empty : 'a dict

  val lookup : 'a dict -> key -> 'a option
  val insert : 'a dict * 'a entry -> 'a dict
end
```

Comment:

Our code for `restoreLeft` and `restoreRight` will be visible *only inside* our structure, *not* to a client.

Specs for `insert` and `ins`

```
(*
  insert : 'a dict * 'a entry -> 'a dict

  REQUIRES: d is a RBT.
  ENSURES:  insert(d,e) is a RBT containing
            exactly all the entries of d
            plus e, with e replacing an entry
            of d if the keys are EQUAL.

  Locally defined helper function ins:

  ins : 'a dict -> 'a dict

  REQUIRES: d is a RBT.
  ENSURES:  ins(d) is a tree containing
            exactly all the entries of d
            plus e, with e replacing an entry
            of d if the keys are EQUAL.

    ins(d) has the same black height as d.

    Moreover, ins(Black(t)) is a RBT
          and ins(Red(t)) is an ARBT.
*)
```

Code for `insert`

```
(* insert : 'a dict * 'a entry -> 'a dict
   REQUIRES and ENSURES RBT.                  *)

fun insert (d , e as (k, v)) =
    let
       fun ins ... (will write shortly)
    in
      (case ins(d) of
         Red(t as (Red(_),_,_)) => Black(t)
       | Red(t as (_,_,Red(_))) => Black(t)
       | d' => d')
    end
```

Code for `insert`

```
(* insert : 'a dict * 'a entry -> 'a dict
   REQUIRES and ENSURES RBT.                *)

fun insert (d , e as (k, v)) =
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        | Red(t as (_,_,Red(_))) => Black(t)
        | d' => d')
    end
```

recall the keyword `as` means
layered pattern matching

Code for `insert`

```
(* insert : 'a dict * 'a entry -> 'a dict
   REQUIRES and ENSURES RBT.                      *)

fun insert (d , e as (k, v)) =
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    in
      (case ins(d) of
         Red(t as (Red(_),_,_)) => Black(t)
        | Red(t as (_,_,Red(_))) => Black(t)
        | d' => d')
    end
```

Here is an acceptable alternate for the `case`:

```
      (case ins(d) of
           Red(t)  => Black(t)
         |   d'    => d')
```

Code for ins

```
(* ins : 'a dict -> 'a dict
   REQUIRES: d is RBT.
   ENSURES: ins(Black(t)) is RBT,
            ins(Red(t)) is ARBT.
   Recall:  e as (k,v)  is in scope.*)

fun ins (Empty) = Red(Empty, e, Empty)
  | ins (Black(ℓ, e' as (k',_), r)) =
    (case String.compare(k,k') of
       EQUAL => Black(ℓ,e,r)  (* replace *)
      |LESS => restoreLeft(Black(ins(ℓ),e',r))
      |_ => restoreRight(Black(ℓ,e',ins(r))))
  | ins (Red(ℓ, e' as (k',_), r)) =
    (case String.compare(k,k') of
       EQUAL => Red(ℓ,e,r)  (* replace *)
      |LESS =>  Red(ins(ℓ),e',r)
      |GREATER => Red(ℓ,e',ins(r)))
```

Code for `ins`

```
(* ins : 'a dict -> 'a dict
   REQUIRES: d is RBT.
   ENSURES: ins(Black(t)) is RBT,
            ins(Red(t)) is ARBT.
   Recall:  e as (k,v)  is in scope.*)

fun ins (Empty) = Red(Empty, e, Empty)
  | ins (Black(ℓ, e′ as (k′,_), r)) =
    (case String.compare(k,k′) of
       EQUAL => Black(ℓ,e,r)   (* replace *)
      |LESS => restoreLeft(Black(ins(ℓ),e′,r))
      |_ => restoreRight(Black(ℓ,e′,ins(r))))
  | ins (Red(ℓ, e′ as (k′,_), r)) =
    (case String.compare(k,k′) of
       EQUAL => Red(ℓ,e,r)   (* replace *)
      |LESS =>  Red(ins(ℓ),e′,r)
      |GREATER => Red(ℓ,e′,ins(r)))
```

Why do we not call `restoreLeft` or `restoreRight` here ?

Code for `lookup`

```
(* lookup : 'a dict -> key -> 'a option *)


fun lookup d k =
    let
        fun lk (Empty) = NONE
          | lk (Red t) = lk' t
          | lk (Black t) = lk' t

        and lk' (ℓ, (k',v), r) =
            (case String.compare(k,k') of
                EQUAL => SOME(v)
              | LESS => lk(ℓ)
              | GREATER => lk(r))
    in
        lk d
    end
```

Code for `lookup`

```
(* lookup : 'a dict -> key -> 'a option *)


fun lookup d k =
    let
        fun lk (Empty) = NONE
          | lk (Red t) = lk' t
          | lk (Black t) = lk' t
        and lk' (ℓ, (k',v), r) =
            (case String.compare(k,k') of
                EQUAL => SOME(v)
              | LESS => lk(ℓ)
              | GREATER => lk(r))
    in
        lk d
    end
```

mutual recursion

Sample Usage

Suppose we have implemented the previous code as:

```
structure RBT :> DICT = struct ... end
```

Now consider:

```
val r1 = RBT.insert(RBT.empty, ("a", 1))
```

Then ML will print:

```
val r1 = - : int RBT.dict
```

because we put in an integer value

because of opaque ascription

Now create the following:

```
val r2 = RBT.insert(r1, ("b", 2))
val look2 = RBT.lookup r2
```

Then `look2 : RBT.key -> int option`

`look2 "a"` $\Longrightarrow$ `SOME 1`

`look2 "c"` $\Longrightarrow$ `NONE`

That is all.

Midterm on Thursday.

Next week we will discuss cost graphs and an abstract datatype designed for writing parallel code (SEQUENCES).