

15–150: Principles of Functional Programming

Some Notes on Regular Expression Matching

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1 Introduction

Regular expression matching is a very useful technique for describing commonly-occurring patterns in strings. For example, the Unix shell (and most other command processors) provides a mechanism (called “globbing”) for describing a collection of files by patterns such as `*.sml` or `hw[1-3].sml`. The emacs text editor provides a richer, but substantially similar, pattern language for its “regexp search” mechanism. In this note we will describe a simple algorithm for regular expression matching that illustrates a number of important programming concepts.

The patterns describable by regular expressions are built up from the following four constructs:

1. *Singleton*: matching a specific character.
2. *Alternation*: choice between two patterns.
3. *Concatenation*: succession of patterns.
4. *Iteration*: indefinite repetition of patterns.

Notably, regular expressions provide no concept of *nesting* of one pattern inside another. For this we would require a richer formalism, such as that provided by *context-free languages*.

2 Languages

To make precise the informal ideas outlined above, we introduce the concept of a *formal language*. First, we fix an *alphabet* Σ , which is any nonempty finite set of *characters* (some authors call these *letters* or *terminals*). The set Σ^* is the set of finite-length *strings* over the alphabet Σ . The *empty string* is written ε . String concatenation is indicated by juxtaposition. A *language* L is any subset of Σ^* — that is, any set of finite-length strings over Σ .

(In this course, in practice, Σ is SML type `char`, and Σ^* is SML type `string`. In this document, we usually will not distinguish between these two types, for simplicity.)

Observe that the empty string ε is represented by the value `""` of type `string` in SML.

*Modified from a draft by Robert Harper.

3 Regular Expressions

Regular expressions are a notation system for languages. The set of regular expressions over an alphabet Σ is given by the following inductive definition (parenthesization is also allowed):

(Notation: Don't confuse the regular expression symbol ε , which refers to the “empty string”, with the set theoretic symbol \in , which means “is in” or “is an element of”.)

Also, $\mathbf{0}$ and $\mathbf{1}$ are special symbols that are not characters in Σ .)

- ▷ If $a \in \Sigma$, then a is a regular expression.
- ▷ $\mathbf{0}$ is a regular expression.
- ▷ $\mathbf{1}$ is a regular expression.
- ▷ If r_1 and r_2 are regular expressions, then so is $r_1 + r_2$.
- ▷ If r_1 and r_2 are regular expressions, then so is $r_1 r_2$.
- ▷ If r is a regular expression, then so is r^* .

The *language* $L(r)$ of a regular expression r is defined as follows:

$$\begin{aligned}
 L(a) &= \{a\} \quad \text{for all } a \in \Sigma \\
 L(\mathbf{0}) &= \{\} \\
 L(\mathbf{1}) &= \{\varepsilon\} \\
 L(r_1 + r_2) &= \{s \mid s \in L(r_1) \text{ or } s \in L(r_2)\} \quad (\text{the “or” is not exclusive}) \\
 L(r_1 r_2) &= \{s_1 s_2 \mid s_1 \in L(r_1) \text{ and } s_2 \in L(r_2)\} \\
 L(r^*) &= \{s_1 \cdots s_n \mid s_i \in L(r) \text{ for } 1 \leq i \leq n, \text{ with } n \geq 0\}
 \end{aligned}$$

By convention, $s_1 \cdots s_n$ stands for the empty string ε if $n = 0$.

We say that a string s *matches* a regular expression r iff $s \in L(r)$. Thus:

- s matches $a \in \Sigma$ iff $s = a$;
- s never matches $\mathbf{0}$;
- s matches $\mathbf{1}$ iff $s = \varepsilon$;
- s matches $r_1 + r_2$ iff s matches either r_1 or r_2 (or both);
- s matches $r_1 r_2$ iff $s = s_1 s_2$, where s_1 matches r_1 and s_2 matches r_2 ;
- s matches r^* iff either $s = \varepsilon$, or $s = s_1 s_2$ where s_1 matches r and s_2 matches r^* .

(The following precedence rules apply when needed: star precedes times precedes plus.)

Some simple examples over the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}\}$:

$$\begin{aligned}
 L(\mathbf{aa}) &= \text{singleton set containing only } \mathbf{aa} \\
 L((\mathbf{a} + \mathbf{b})^*) &= \text{set of all strings over } \Sigma \\
 L((\mathbf{a} + \mathbf{b})^* \mathbf{aa} (\mathbf{a} + \mathbf{b})^*) &= \text{set of strings with two consecutive } \mathbf{a}'\text{s} \\
 L((\mathbf{a} + \mathbf{1})(\mathbf{b} + \mathbf{ba})^*) &= \text{set of strings over } \Sigma \text{ without two consecutive } \mathbf{a}'\text{s}
 \end{aligned}$$

4 A Matching Algorithm

We are to define an SML function `accept` with type `regexp -> string -> bool` such that `accept r s` evaluates to `true` iff `s` matches `r`, and evaluates to `false` otherwise.

First we require a representation of regular expressions in SML:

```
datatype regexp = Char of char
                | Zero
                | One
                | Plus of regexp * regexp
                | Times of regexp * regexp
                | Star of regexp
```

The correspondence to the definition of regular expressions should be clear.

We will implement the matcher using continuations with Boolean return type. We will define an auxiliary function `match` of type

```
match : regexp -> char list -> (char list -> bool) -> bool
```

which takes a regular expression, a character list, and a continuation, and yields a Boolean value. Informally, the continuation determines how to proceed once an initial segment of the given character list has been determined to match the given regular expression — the remaining input is passed to the continuation to determine the final outcome. In order to ensure that the matcher succeeds (yields `true`) whenever possible, we must be sure to consider *all* ways in which an initial segment of the input character list matches the given regular expression in such a way that the remaining unmatched input causes the continuation to succeed. Only if there is no way to do so may we yield `false`.

This informal specification may be made precise as follows:

(Here we are treating the arguments `s`, `s1`, and `s2` as strings for conceptual simplicity, even though in our implementation they will be character lists, exploded from those strings.)

1. If there exists `s1` and `s2` such that `s = s1 s2`, `s1 ∈ L(r)`, and `k(s2)` evaluates to `true`, then `match r s k` evaluates to `true`.
2. If for every `s1` and `s2` such that `s = s1 s2` with `s1 ∈ L(r)` we have that `k(s2)` evaluates to `false`, then `match r s k` evaluates to `false`

Notice that this specification determines the outcome only for continuations `k` that always yield either `true` or `false` on any input. This is sufficient for our purposes since the continuations that arise will always satisfy this requirement. (That fact is by no means obvious and requires proof. We will however omit the proof in this document.)

Notice as well that the specification implies that the result should be `false` in the case that there is no way to partition the input string `s` such that an initial segment matches `r`.

Before giving the implementation of `match`, we can define `accept` as follows:

```
fun accept r s = match r (String.explode s) List.null
```

Two remarks. We “explode” the string argument into a list of characters to facilitate sequential processing of the string. The *initial continuation* yields `true` or `false` according to whether the remaining input has been exhausted. Assuming that `match` satisfies the specification given above, one sees that `accept` is indeed the required matching algorithm.

Now for the code for `match`:

```
(* match : regexp -> char list -> (char list -> bool) -> bool *)
fun match (Char(a)) cs k = (case cs of
    nil => false
  | (c::cs') => (a=c) andalso (k cs'))
| match (Zero) _ _ = false
| match (One) cs k = k cs
| match (Plus(r1,r2)) cs k = match r1 cs k orelse match r2 cs k
| match (Times(r1,r2)) cs k = match r1 cs (fn cs' => match r2 cs' k)
| match (Star(r)) cs k = k cs orelse match r cs (fn cs' => match (Star(r)) cs' k)
```

Note that the case of `(Star r)` could have been written

```
| match (Star(r)) cs k =
    match (Plus (One, Times (r, (Star r)))) cs k
```

but this has the disadvantage of creating a new regular expression during matching.

Does `match` satisfy the specification given above? A natural way to approach the proof is to proceed by induction on the structure of the regular expression. We examine here an abbreviated informal version of that proof for two cases. First, consider the case $r = \text{Times}(r_1, r_2)$. We have two proof obligations according to whether or not the input may be partitioned in such a way that an initial segment matches r and the continuation succeeds on the corresponding final segment.

First, suppose that $s = s_1 s_2$ with s_1 matching r and $k(s_2)$ evaluates to `true`. We are to show that `match r s k` evaluates to `true`. Now since s_1 matches r , we have that $s_1 = s_{1,1} s_{1,2}$ with $s_{1,1}$ matching r_1 and $s_{1,2}$ matching r_2 . Consequently, by the induction hypothesis applied to r_2 , we have that `match r_2 (s_{1,2} s_2) k` evaluates to `true`. Therefore the application `(fn cs' => match r_2 cs' k) (s_{1,2} s_2)` evaluates to `true`, and hence by the induction hypothesis applied to r_1 , the expression `match r_1 s (fn cs' => match r_2 cs' k)` evaluates to `true`, which is enough for the result.

Second, suppose that no matter how we choose s_1 and s_2 such that $s = s_1 s_2$ with $s_1 \in L(r)$, we have that $k(s_2)$ evaluates to `false`. It suffices to show that `match r_1 s (fn cs' => match r_2 cs' k)` evaluates to `false`. By the induction hypothesis (applied to r_1) it suffices to show that for every $s_{1,1}$ and s'_2 such that $s = s_{1,1} s'_2$ with $s_{1,1} \in L(r_1)$, we have that `match r_2 s'_2 k` evaluates to `false`. By the induction hypothesis (applied to r_2) it suffices to show that for every $s_{1,2}$ and s_2 such that $s'_2 = s_{1,2} s_2$ with $s_{1,2} \in L(r_2)$, we have that $k(s_2)$ evaluates to `false`. But this follows from our assumptions, taking $s_1 = s_{1,1} s_{1,2}$.

This completes the proof for $r = r_1 r_2$; you should carry out the proof for $a \in \Sigma$, $\mathbf{1}$, $\mathbf{0}$, and $r_1 + r_2$.

What about iteration? Let r be `Star r_1`, and suppose that $s = s_1 s_2$ with s_1 matching r and $k(s_2)$ evaluates to `true`. By our choice of r , there are two cases to consider: either $s_1 = \epsilon$, or $s_1 = s_{1,1} s_{1,2}$ with $s_{1,1}$ matching r_1 and $s_{1,2}$ matching r . In the former case, the result is the result of $k(s)$, which is $k(s_2)$, which is `true`, as required. In the latter case, it suffices to show that `match r_1 s (fn cs' => match r cs' k)` evaluates to `true`. By the induction hypothesis, it suffices to show that `match r (s_{1,2} s_2) k` evaluates to `true`. It is tempting at this stage to appeal to the induction hypothesis to complete the proof — but we cannot because the regular expression argument is the *original* regular expression r , and not some sub-expression of it!

What to do? In general there are two possibilities: fix the proof or find a counterexample to the theorem. Let's try to fix the proof — it will help us to find the counterexample (the theorem as

stated, i.e., that the matcher behaves correctly on `Star`, is false). A natural attempt is to proceed by an “outer” induction on the structure of the regular expression (as we’ve done so far), together with an “inner” induction on the length of the string, the idea being that in the case of iteration we appeal to the *inner* induction hypothesis to complete the proof — provided that the string $s_{1,2} s_2$ can be shown to be shorter than the string s . This is equivalent to showing that $s_{1,1}$ is non-empty — but that could be false! For example, r might be 1^* , in which case our matcher loops forever.

What to do? Change the specification. The proof goes through provided that every regular expression is in *standard form*, by which we mean that if r^* occurs within a regular expression, then the empty string is not in the language of r . Given this condition, the proof goes through as outlined. You should carry out the proof for the case of iteration to check this claim. Remember that there are two parts to the proof: complete the proof for the case that there is a partitioning that leads to the continuation yielding `true` (sketched above), and for the case that there is no such partitioning. Consult the proof for the concatenation of regular expressions as a guide.

In the next section we will see how to convert any regular expression into standard form, so we can apply the algorithm as given above.

Another solution is to rule out matches of r against the empty string when matching r^* against any string. This never rules out a valid solution, but we can use the idea to force termination (which is the only the problem with the algorithm above). All cases remain the same, except for the case of `Star(r)`.

```
| match (Star(r)) cs k =
  k cs orelse
  match r cs (fn cs' => properSuffix(cs',cs) andalso match (Star(r)) cs' k)
```

Here `properSuffix(cs',cs)` returns `true` if cs' is a proper suffix of cs and `false` otherwise. We can prove inductively that cs' is a suffix of cs , so if `properSuffix(cs',cs)` returns `false` then cs' is actually identical to cs . That means the initial segment of cs matched by r must have been empty.

The correctness proof now works by two nested structural inductions: one on the structure of the regular expression r and one on the structure of the string we match against. This means that either (a) the regular expression has to get smaller (which it does in all cases except within the continuation of the `Star` case), or (b) if the regular expression stays the same, then the string has to get smaller. One can see that this is guaranteed in the modified program above.

5 Standardization

If we do not want to change the program, we can restrict the input to be in standard form. But have we lost something by making this restriction? Are there *non-standard* regular expressions of interest? What have we left out? The answer is nothing, because *any regular expression can be brought into standard form!* The idea is that before matching commences we rewrite the regular expression into another regular expression defining the same language and in standard form.

Here is how it is done. We rely on the equation $L(r) = L(\delta(r) + r^-)$, where $\delta(r)$ is either **1** or **0** according to whether or not $L(r)$ contains the empty string, and where r^- is defined so that $L(r^-) = L(r) \setminus \{\varepsilon\}$.

The functions $\delta(r)$ and r^- are defined as follows.

$$\begin{aligned} \delta(\mathbf{0}) &= \mathbf{0} \\ \delta(\mathbf{1}) &= \mathbf{1} \\ \delta(\mathbf{a}) &= \mathbf{0} \\ \delta(r_1 + r_2) &= \delta(r_1) \oplus \delta(r_2) \\ \delta(r_1 r_2) &= \delta(r_1) \otimes \delta(r_2) \\ \delta(r^*) &= \mathbf{1} \end{aligned}$$

We define $r_1 \oplus r_2$ to be **1** if either r_1 or r_2 is **1**, and **0** otherwise. We define $r_1 \otimes r_2$ to be **0** if either r_1 or r_2 is **0**, and **1** otherwise.

Finally, we define r^- as follows:

$$\begin{aligned} \mathbf{0}^- &= \mathbf{0} \\ \mathbf{1}^- &= \mathbf{0} \\ \mathbf{a}^- &= \mathbf{a} \\ (r_1 + r_2)^- &= r_1^- + r_2^- \\ (r_1 r_2)^- &= \delta(r_1) r_2^- + r_1^- \delta(r_2) + r_1^- r_2^- \\ (r^*)^- &= r^-(r^-)^* \end{aligned}$$

The last two deserve comment. The non-empty strings matching $r_1 r_2$ are (1) the non-empty strings in r_2 , in the case that r_1 contains the empty string, (2) the non-empty strings in r_1 , in the case that r_2 contains the empty string, and (3) the concatenation of a non-empty string in r_1 followed by a non-empty string in r_2 . Check that the given equation expresses these three conditions as a regular expression! The clause for iteration is motivated by the observation that the non-empty strings in the iteration r^* are simple the *non-zero* iterations of the *non-empty* strings in r .

As an exercise, prove that $\delta(r)$ and r^- have the required properties (stated above), that $\delta(r) + r^-$ is in standard form, and that $L(r) = L(\delta(r) + r^-)$. We thus can replace r by $\delta(r) + r^-$ without losing any matching ability and without needing to change our code from the first implementation of `match` given on page 4.

6 Conclusion

The example of regular expression matching illustrates a number of important programming concepts:

1. *Continuations*: the use of continuations to perform interesting search.
2. *Proof-directed debugging*: the use of a breakdown in a proof attempt to discover an error in the code.
3. *Change of specification*: once we isolated the error, we had the choice of changing the *code* or changing the *specification*. Moral: debugging isn't always a matter of hacking!
4. *Pre-processing*: to satisfy the more stringent specification we pre-processed the regular expression so that it satisfied the additional assumption required for correctness, without losing the generality of the matcher.