Lists, structural induction, tail recursion, and extensional equivalence

15-150

Lecture 4: September 3, 2026

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Taking stock

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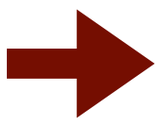

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more on datatypes in later lectures!

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  - today: example using **structural induction** on lists

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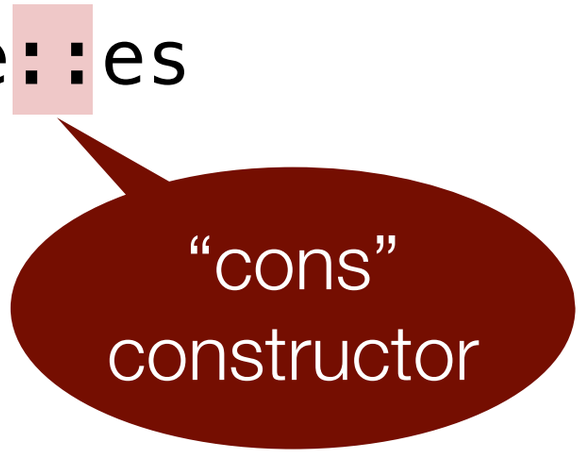

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"cons" constructor

can be matched against!

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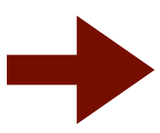

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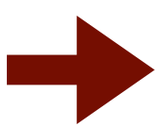

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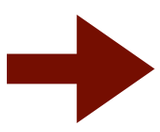

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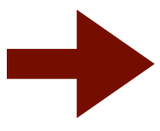

cons evaluates **left to right** (for sequential evaluation)

Length function for an int list

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(* length : int list -> int
   REQUIRES: true
   ENSURES: length(L) returns the number of
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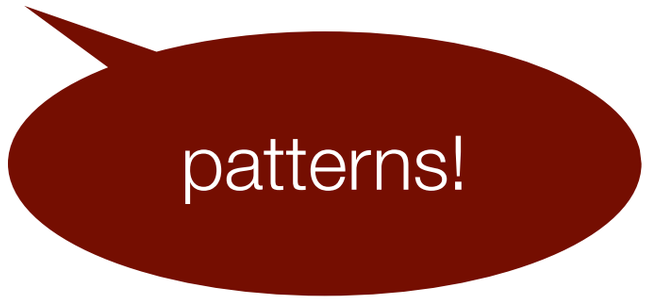

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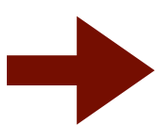

Let’s verify that length is **total** (“always yields a value”)!

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- Unlike in math, functions in SML are not total
  - eg, recursive functions can loop!

Totality and functions

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→ Partiality complicates extensional equivalence (more later!)

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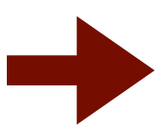

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- We proceed by **structural induction** on argument list.

Structural induction for lists

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To prove a property P(L) for every value L of type int list:

- show that P([ ]) holds
- show that, if P(L') for some value L' of type int list,
  then it also holds for v :: L' (for any value v of type int).

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- We proceed by structural induction on argument list.
  - Let’s do the proof together!

Length is total: proof

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Theorem: For al values `L:int list`, `length(L)` evaluates to an integer value `v`.

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Base case: L = [].

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Showing:

```
    length []
⟹  0                                   (step, 1st clause of length)
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Showing:

```
    length(x::xs)
⟹  1 + length(xs)          (step, 2nd clause of length)
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⟹  1 + v                 (IH)
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Showing:

	`length(x::xs)`	
$\Longrightarrow$	`1 + length(xs)`	(step, 2nd clause of `length`)
$\Longrightarrow$	`1 + v`	(IH)
$\Longrightarrow$	`v'`	(some `v'`, assume `+` total)

Length is total: proof

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fun length ([] : int list) : int = 0
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⟹  1 + length(xs)        (step, 2nd clause of length)
⟹  1 + v                 (IH)
⟹  v'                    (some v', assume + total)
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SML addition must be total!

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base case(s):		
inductive/recursive case(s):		

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`0`	`fun power(_, 0) = 1`	`power(_, 0)` ↪ 1
inductive/recursive case(s):		
`(k-1)+1`	`power(n,k) = n*power(n, k-1)`	IH: `power(n,k)` ↪ n^k NTS: `power(n,k+1)` ↪ n^{k+1}

Correspondence

Data structure	Code	Proof
base case(s):		
`0`	`fun power(_, 0) = 1`	`power(_, 0)` ↪ 1
`[ ]`	`0`	`length []` ↪ 0
inductive/recursive case(s):		
`(k-1)+1`	`power(n,k) = n*power(n, k-1)`	IH: `power(n,k)` ↪ n^k NTS: `power(n,k+1)` ↪ n^{k+1}
`x :: xs`	`length(x::xs) = 1+length(xs)`	IH: `length(xs)` ↪ v NTS: `length(x::xs)` ↪ v’

Correspondence

Data structure	Code	Proof
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`[ ]`	`0`	`length []` ↪ 0
inductive/recursive case(s):		
`(k-1)+1`	`power(n,k) = n*power(n, k-1)`	IH: `power(n,k)` ↪ n^k NTS: `power(n,k+1)` ↪ n^{k+1}
`x :: xs`	`length(x::xs) = 1+length(xs)`	IH: `length(xs)` ↪ v NTS: `length(x::xs)` ↪ v'

→ possibly several bases and inductive cases!

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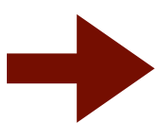

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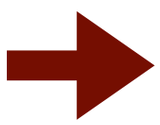

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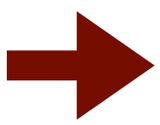

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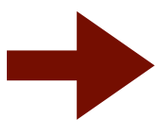

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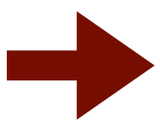

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```

[3/x, [4,5]/xs] 1 + length(xs)

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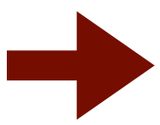

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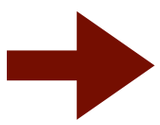

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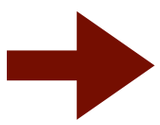

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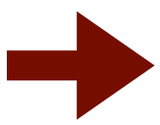

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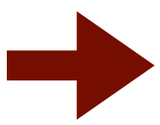

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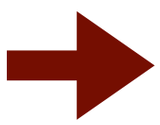

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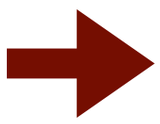

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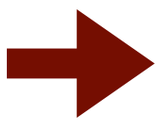

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time

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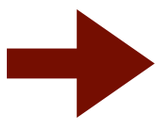

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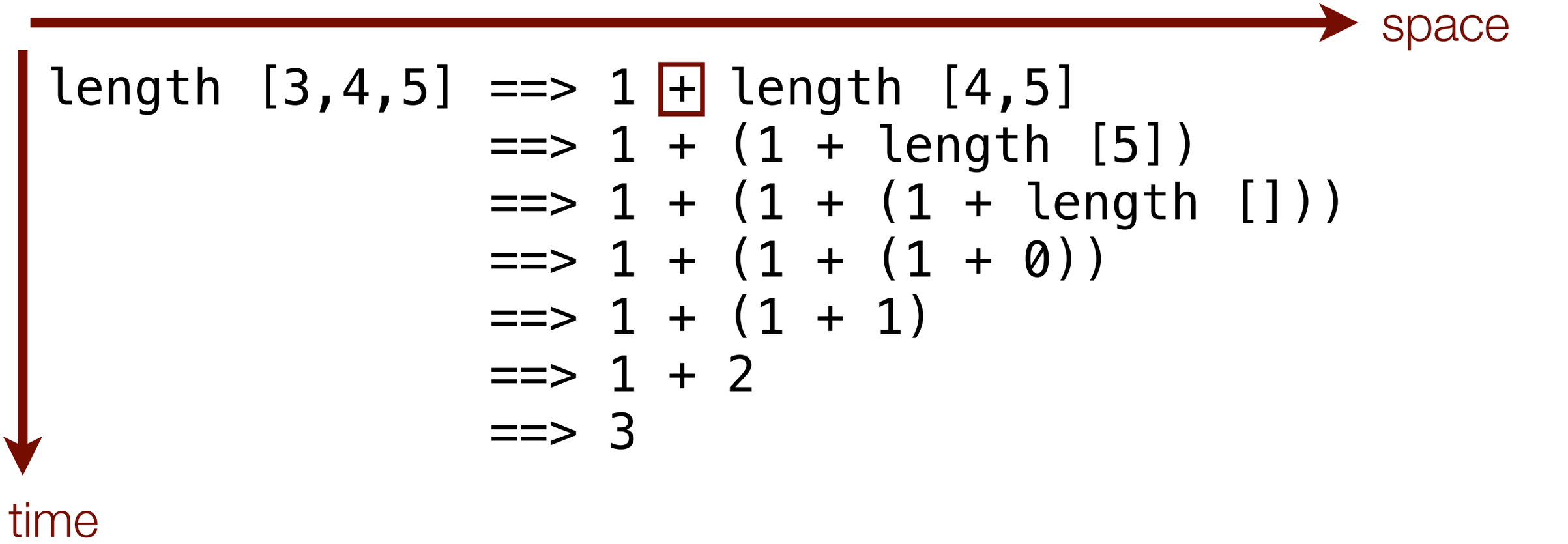

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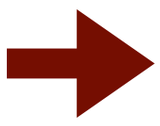

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→ Is this function efficient? Well, let's look at an evaluation trace:

space →

time ↓

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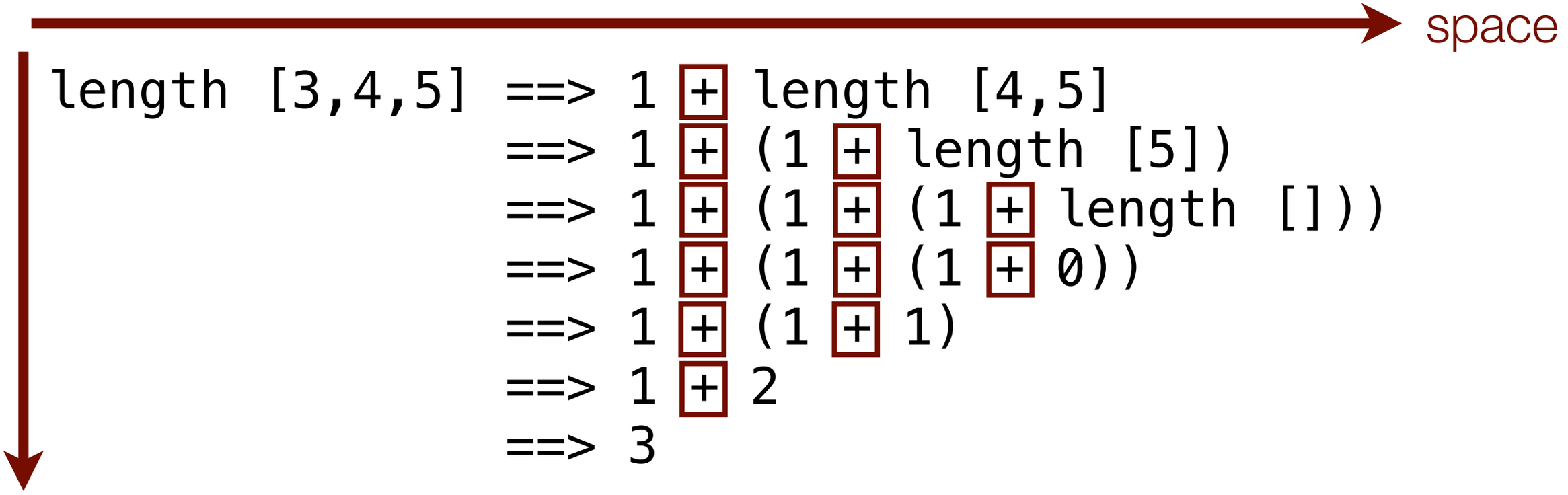

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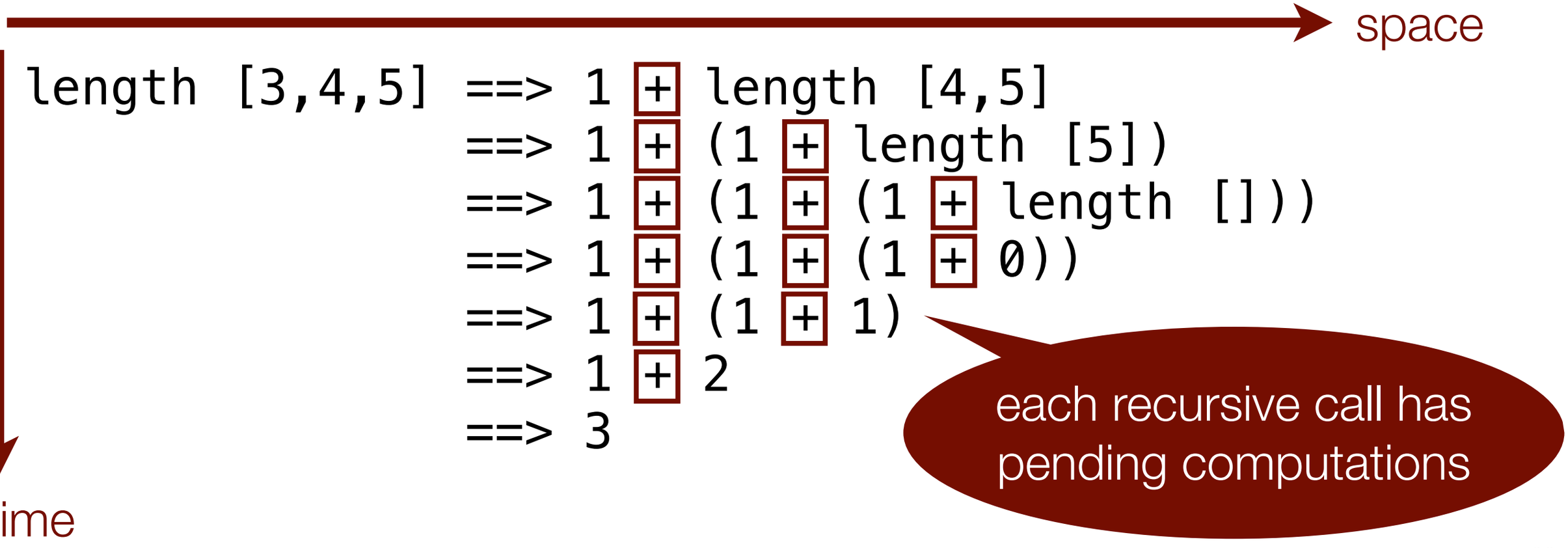

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accumulator

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extensional equivalence

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space-inefficient version

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tail call

`1+acc` happens before recursive call!

Remember: evaluation order for applications

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Function application: $\mathtt{e}_1 \ \ \mathtt{e}_2$

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Remember: evaluation order for applications

Function application: e_1 e_2

1. Evaluate e_1 to a function `value fn (x:t) => e`.
2. Evaluate e_2 to a value `v`.
3. Evaluate function body `e` with `v` substituted for `x`, i.e., `[v/x]e`.
   - i. For a recursive function `fun f x = e`, substitute function definition itself for `f` in the function body.*

* See lecture notes of Week 2 for further details.

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Improved space efficiency

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```
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                ==> tlength ([4,5], 1+0)
                ==> tlength ([4,5], 1)
                ==> tlength ([5], 1+1)
                ==> tlength ([5], 2)
                ==> tlength ([], 2+1)
                ==> tlength ([], 3)
                ==> 3
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time

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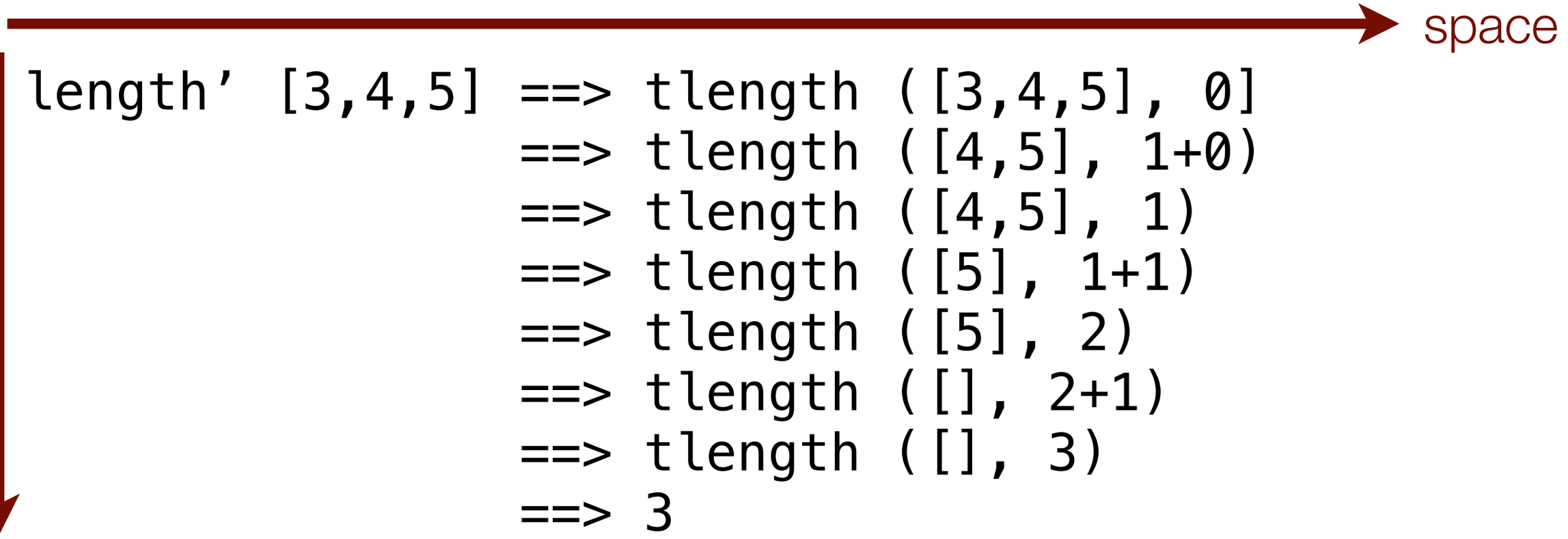


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space

time

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```

recursive calls without pending computations

Let’s prove space-efficient version correct!

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Theorem: For all values `L: int list` and `acc: int`, `tlength(L,acc)` ≅ `(length L) + acc`.

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→ Before doing the proof, let's review extensional equivalence!

Extensional equivalence, revisited

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Definition:

- **Expression** of type `int` are **extensionally equivalent**,
  - if they evaluate to the same integer, or
  - if they both loop forever, or
  - if they both raise the same exception.
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Facts:

- If $e_1 \hookrightarrow v$ and $e_2 \hookrightarrow v$, then $e_1 \cong e_2$.
- If $e_1 \Longrightarrow e$ and $e_2 \Longrightarrow e$, then $e_1 \cong e_2$.
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From $e_1 \cong e_2$ it does **not** necessarily follow that $e_1 \Longrightarrow e_2$ or $e_2 \Longrightarrow e_1$!

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eg, $1+1+7 \cong 9$

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Moreover:

For f: `int -> int`, it does **not** necessarily follow that `f(1) + f(2)` $\cong$ `f(2) + f(1)`!

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eg, `1+1+7` $\cong$ `9`

f may loop or raise an exception (on some inputs)

Moreover:

For f: `int -> int`, it does **not** necessarily follow that `f(1) + f(2)` $\cong$ `f(2) + f(1)`!

Let’s prove space-efficient version correct!

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fun length ([] : int list) : int = 0
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Proof: By ???

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Proof: By structural induction on `L`.

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Proof: By structural induction on L.

Base case: $L = [\,]$.

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Proof: By structural induction on L.

Base case: L = `[]`.

Need to show: for all values `acc:int`,

`tlength([],acc)` ≅ `length ([]) + acc`.

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⟹ `acc` (1st clause of `tlength`)

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Evaluating the left side:

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Evaluating the right side:

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    length([]) + acc
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→ equivalent because both reduce to the same value.

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   tlength(x::xs, acc)
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because: if $e_1 \Longrightarrow e_2$, then $e_1 \cong e_2$

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Showing:

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≅	`tlength(xs, 1 + acc)`	(step, 2nd clause of `tlength`)
≅	`length(xs) + (1 + acc)`	(IH, assume `+` total)
≅	`(1 + length(xs)) + acc`	(`+` commutative, associative, totality of `length`)
≅	`length(x::xs) + acc`	(step, 2nd clause of `length`)

→ Proved by structural induction on lists.

Appending lists

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(* append: int list * int list -> int list
   REQUIRES: true
   ENSURES: append(L,R) evaluates to a list consisting
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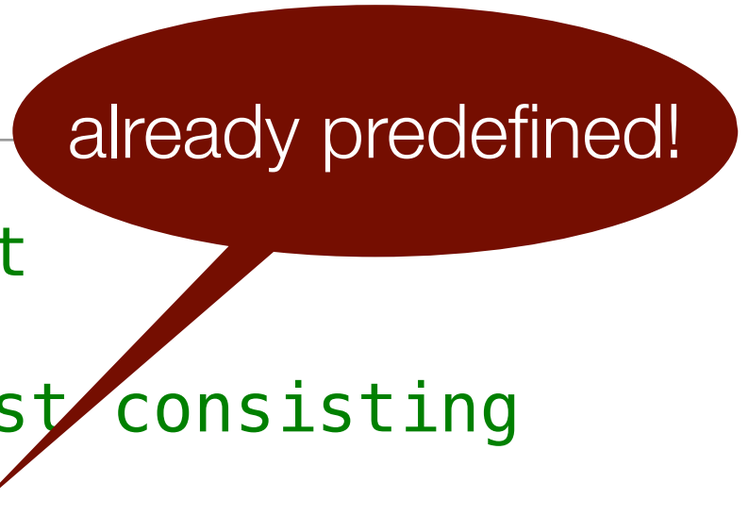


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- What is the time complexity of append?
- append(X,Y) has time complexity O(|X|).

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- reverse(X) has time complexity $O(|X|^2)$.

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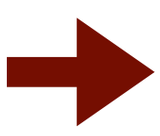

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Theorem: For all values `L: int list` and `acc: int list`, `trev(L,acc)` ≅ `(rev L) @ acc`.

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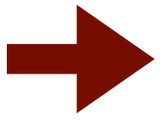

Prove this theorem as an exercise!

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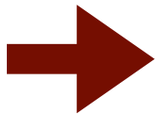

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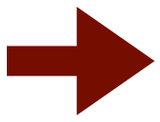

We provide the solution in the notes (rev.pdf). But try first!

That's all for today. Have a good weekend!