

Recursion and Induction

15-150

Lecture 3: September 2, 2025

Stephanie Balzer
Carnegie Mellon University

Recap of week 1

Recap of week 1

Functional programming

Recap of week 1

Functional programming

➔ evaluation of expressions (no mutation!)

Recap of week 1

Functional programming

- ➔ evaluation of expressions (no mutation!)
- ➔ facilitates specification and reasoning about program

Recap of week 1

Functional programming

- evaluation of expressions (no mutation!)
- facilitates specification and reasoning about program
- correctness proof (today's topic!)

Recap of week 1

Functional programming

- ➔ evaluation of expressions (no mutation!)
- ➔ facilitates specification and reasoning about program
 - ➔ correctness proof (today's topic!)
- ➔ facilitates parallelism

Recap of week 1

Functional programming

- ➔ evaluation of expressions (no mutation!)
- ➔ facilitates specification and reasoning about program
 - ➔ correctness proof (today's topic!)
- ➔ facilitates parallelism

Types, expressions, values

Recap of week 1

Functional programming

- ➔ evaluation of expressions (no mutation!)
- ➔ facilitates specification and reasoning about program
 - ➔ correctness proof (today's topic!)
- ➔ facilitates parallelism

Types, expressions, values

- ➔ types as specifications

Recap of week 1

Functional programming

- ➔ evaluation of expressions (no mutation!)
- ➔ facilitates specification and reasoning about program
 - ➔ correctness proof (today's topic!)
- ➔ facilitates parallelism

Types, expressions, values

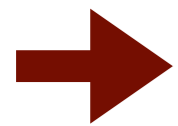
- ➔ types as specifications
- ➔ observation: once your program type checks, it works!

Recap of week 1

Extensional equivalence (\cong)

Recap of week 1

Extensional equivalence (\cong)



“Two things are equal if they behave the same”

Recap of week 1

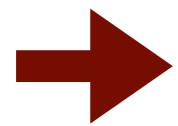
we'll revisit
exact definition

Extensional equivalence (\cong)

➔ “Two things are equal if they behave the same”

Recap of week 1

Extensional equivalence (\cong)



“Two things are equal if they behave the same”

Recap of week 1

Extensional equivalence (\cong)

- ➔ “Two things are equal if they behave the same”
- ➔ facilitates compositional (aka modular) reasoning

Recap of week 1

Extensional equivalence (\cong)

- “Two things are equal if they behave the same”
- facilitates compositional (aka modular) reasoning
 - replace equals by equals in any sub-expression

Recap of week 1

Extensional equivalence (\cong)

- ➔ “Two things are equal if they behave the same”
- ➔ facilitates compositional (aka modular) reasoning
 - ➔ replace equals by equals in any sub-expression

Declarations, binding and scope

Recap of week 1

Extensional equivalence (\cong)

- “Two things are equal if they behave the same”
- facilitates compositional (aka modular) reasoning
 - replace equals by equals in any sub-expression

Declarations, binding and scope

- shadowing of bindings

Recap of week 1

Extensional equivalence (\cong)

- “Two things are equal if they behave the same”
- facilitates compositional (aka modular) reasoning
 - replace equals by equals in any sub-expression

Declarations, binding and scope

- shadowing of bindings
- function declarations bind a closure to the function identifier

Recap of week 1

Extensional equivalence (\cong)

- “Two things are equal if they behave the same”
- facilitates compositional (aka modular) reasoning
 - replace equals by equals in any sub-expression

Declarations, binding and scope

- shadowing of bindings
- function declarations bind a closure to the function identifier
 - closure comprises lambda expression and environment with bindings existing at declaration time

Recap of week 1

Pattern matching

Recap of week 1

Pattern matching

➔ patterns are used at binding sites of values

Recap of week 1

Pattern matching

- patterns are used at binding sites of values
 - eg, val bindings, function arguments, case expression

Recap of week 1

Pattern matching

- patterns are used at binding sites of values
- eg, val bindings, function arguments, case expression
- allow us to match against an expected value

Recap of week 1

Pattern matching

- patterns are used at binding sites of values
 - eg, val bindings, function arguments, case expression
- allow us to match against an expected value
- allow us to decompose a value in its constituent parts, introducing appropriate bindings for parts

Recap of week 1

5-step methodology of function declaration

Recap of week 1

5-step methodology of function declaration

1 function name and type

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
- 3 ENSURES: postcondition

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
- 3 ENSURES: postcondition
- 4 function body

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
- 3 ENSURES: postcondition
- 4 function body
- 5 tests

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
- 3 ENSURES: postcondition
- 4 function body
- 5 tests

Today, we add a 6th step:

Recap of week 1

5-step methodology of function declaration

- 1 function name and type
- 2 REQUIRES: precondition
- 3 ENSURES: postcondition
- 4 function body
- 5 tests

Today, we add a 6th step:

- 6 correctness proof

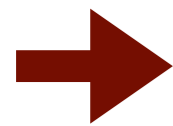
Today's topic: functional correctness

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!



we will use three kinds of induction:

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

→ we will use three kinds of induction:

→ mathematical induction

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

→ we will use three kinds of induction:

→ mathematical induction

→ strong induction

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

→ we will use three kinds of induction:

→ mathematical induction

→ strong induction

→ structural induction

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

→ we will use three kinds of induction:

→ mathematical induction

→ strong induction

→ structural induction

→ we consider how expressions are evaluated

Today's topic: functional correctness

Let's prove our programs correct, one function at a time!

- we will use three kinds of induction:
 - mathematical induction
 - strong induction
 - structural induction
- we consider how expressions are evaluated
- we may appeal to mathematical properties and assume that SML implements them correctly

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
| power (n:int, k:int) : int = n * power(n, k-1)
```

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
| power (n:int, k:int) : int = n * power(n, k-1)
```



pattern matching

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```



recursive call

An example: compute n^k

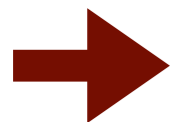
```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```



this function is not very efficient:

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

➔ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

➔ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

➔ Number of recursive calls: $O(k)$

An example: compute n^k

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)
```

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

➔ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

➔ Number of recursive calls: $O(k)$

➔ Can we do better than that?

Idea for making power more efficient

→ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

→ Number of recursive calls: $O(k)$

→ Can we do better than that?

Idea for making power more efficient

→ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

→ Number of recursive calls: $O(k)$

→ Can we do better than that?

Assume we have functions **even** and **square**. Now we can get a more efficient implementation:

Idea for making power more efficient

→ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

→ Number of recursive calls: $O(k)$

→ Can we do better than that?

Assume we have functions **even** and **square**. Now we can get a more efficient implementation:

$$\text{eg, } 3^7 = 3 * (3^3)^2$$

Idea for making power more efficient

→ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

→ Number of recursive calls: $O(k)$

→ Can we do better than that?

Assume we have functions **even** and **square**. Now we can get a more efficient implementation:

$$\begin{aligned}\text{eg, } 3^7 &= 3 * (3^3)^2 \\ &= 3 * (3 * (3^1)^2)^2\end{aligned}$$

Idea for making power more efficient

→ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

→ Number of recursive calls: $O(k)$

→ Can we do better than that?

Assume we have functions **even** and **square**. Now we can get a more efficient implementation:

$$\begin{aligned}\text{eg, } 3^7 &= 3 * (3^3)^2 \\ &= 3 * (3 * (3^1)^2)^2 \\ &= 3 * (3 * (3 * 1)^2)^2\end{aligned}$$

Idea for making power more efficient

➔ this function is not very efficient:

$$\text{eg, } 3^7 = 3 * 3 * 3 * 3 * 3 * 3 * 3$$

➔ Number of recursive calls: $O(k)$

➔ Can we do better than that?

Assume we have functions **even** and **square**. Now we can get a more efficient implementation:

$$\begin{aligned}\text{eg, } 3^7 &= 3 * (3^3)^2 \\ &= 3 * (3 * (3^1)^2)^2 \\ &= 3 * (3 * (3 * 1)^2)^2\end{aligned}$$

➔ Number of recursive calls: $O(\log(k))$

A more efficient version of power

A more efficient version of power

```
(* even : int -> bool
   REQUIRES: true
   ENSURES: even(k) evaluates to true if k is even
              evaluates to false if k is odd.
*)
```

A more efficient version of power

```
(* even : int -> bool
   REQUIRES: true
   ENSURES: even(k) evaluates to true if k is even
             evaluates to false if k is odd.
*)
```

```
fun even (k:int) : bool = ((k mod 2) = 0)
```

A more efficient version of power

```
(* even : int -> bool
   REQUIRES: true
   ENSURES: even(k) evaluates to true if k is even
              evaluates to false if k is odd.
*)
```

```
fun even (k:int) : bool = ((k mod 2) = 0)
```

```
(* square : int -> int
   REQUIRES: true
   ENSURES: square(n) ==> n^2
*)
```

A more efficient version of power

```
(* even : int -> bool
   REQUIRES: true
   ENSURES: even(k) evaluates to true if k is even
             evaluates to false if k is odd.
*)
```

```
fun even (k:int) : bool = ((k mod 2) = 0)
```

```
(* square : int -> int
   REQUIRES: true
   ENSURES: square(n) ==> n^2
*)
```

```
fun square (n:int) : int = n * n
```

A more efficient version of power

(* powere : (int * int) -> int

REQUIRES: $k \geq 0$

ENSURES: $\text{powere}(n, k) \Rightarrow n^k$, with $0^0 = 1$.

powere computes n^k using $O(\log(k))$ multiplies.

*)

A more efficient version of power

```
(* powere : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: powere(n,k) ==> n^k, with 0^0 = 1.

   powere computes n^k using O(log(k)) multiplies.
*)
```

```
fun powere (_:int, 0:int) : int =
  | powere (n:int, k:int) : int =
```

A more efficient version of power

```
(* powere : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: powere(n,k) ==> n^k, with 0^0 = 1.

   powere computes n^k using O(log(k)) multiplies.
*)
```

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
```

A more efficient version of power

```
(* powere : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: powere(n,k) ==> n^k, with 0^0 = 1.

   powere computes n^k using O(log(k)) multiplies.
*)
```

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
```

```
  if even(k)
  then square(powere(n, k div 2))
```

exponent k is even

A more efficient version of power

```
(* powere : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: powere(n,k) ==> n^k, with 0^0 = 1.

   powere computes n^k using O(log(k)) multiplies.
*)
```

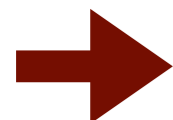
```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

A more efficient version of power

```
(* powere : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: powere(n,k) ==> n^k, with 0^0 = 1.

   powere computes n^k using O(log(k)) multiplies.
*)
```

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```



Number of recursive calls: $O(\log(k))$

Let's verify our naive version of power

Let's verify our naive version of power

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)

fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Let's verify our naive version of power

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)

fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```



How shall we proceed?

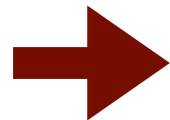
Let's verify our naive version of power

```
(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
*)

fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```



How shall we proceed?



Let's use mathematical induction!

Mathematical (simple, weak) induction

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.



base case

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$, $P(k+1)$ follows logically from $P(k)$.



inductive step

Mathematical (simple, weak) induction

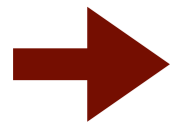
To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.

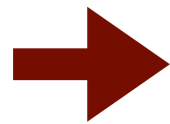


Why does it work?

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.



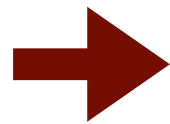
Why does it work?

- $P(0)$ is proved directly.

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.



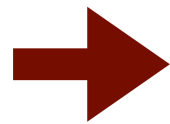
Why does it work?

- $P(0)$ is proved directly.
- $P(1)$ follows from $P(0)$.

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.



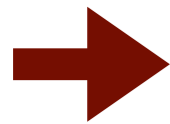
Why does it work?

- $P(0)$ is proved directly.
- $P(1)$ follows from $P(0)$.
- $P(2)$ follows from $P(1)$.

Mathematical (simple, weak) induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k \geq 0$,
 $P(k+1)$ follows logically from $P(k)$.



Why does it work?

- $P(0)$ is proved directly.
- $P(1)$ follows from $P(0)$.
- $P(2)$ follows from $P(1)$.
- etc...

Let's verify our naive version of power

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



Proof by mathematical induction on ???

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



Proof by mathematical induction on k .

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

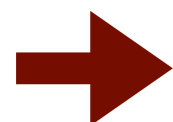
→ Proof by mathematical induction on k .

k is the integer that gets smaller!

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



Proof by mathematical induction on k .

k is the integer that gets smaller!

needed for applying IH!

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

➔ Proof by mathematical induction on k .

➔ Let's do the proof together!

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Base case: $k = 0$.

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Base case: $k = 0$.

Need to show: `power(n, 0)` evaluates to n^0 , for all n . Note: $n^0 = 1$.

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Base case: $k = 0$.

Need to show: `power(n, 0)` evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Base case: $k = 0$.

Need to show: `power(n, 0)` evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

`power(n, 0)`

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Base case: $k = 0$.

Need to show: `power(n, 0)` evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

`power(n, 0)`
 $\Rightarrow 1$

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Theorem: `power(n, k)` evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By mathematical induction on k .

Base case: $k = 0$.

Need to show: `power(n, 0)` evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

`power(n, 0)`
 $\Rightarrow 1$ (step, 1st clause of `power`)

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: `power(n, k)` evaluates to n^k , for $k \geq 0$ and all integers n .

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: `power(n, k)` evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: `power(n, k+1)` evaluates to n^{k+1} .

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: `power(n, k)` evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: `power(n, k+1)` evaluates to n^{k+1} .

Showing:

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: `power(n, k)` evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: `power(n, k+1)` evaluates to n^{k+1} .

Showing:

`power(n, k+1)`

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$

$\Rightarrow n * \text{power}(n, k+1-1)$

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$
 $\implies n * \text{power}(n, k+1-1)$ (step, 2nd clause of `power`)

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$

$\Rightarrow n * \text{power}(n, k+1-1)$ (step, 2nd clause of power)

$\Rightarrow n * \text{power}(n, k)$

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$

$\implies n * \text{power}(n, k+1-1)$ (step, 2nd clause of `power`)

$\implies n * \text{power}(n, k)$ (math)

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$

$\Rightarrow n * \text{power}(n, k+1-1)$ (step, 2nd clause of `power`)

$\Rightarrow n * \text{power}(n, k)$ (math)

$\Rightarrow n * n^k$

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$	
$\implies n * \text{power}(n, k+1-1)$	(step, 2nd clause of <code>power</code>)
$\implies n * \text{power}(n, k)$	(math)
$\implies n * n^k$	(IH)

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$	
$\implies n * \text{power}(n, k+1-1)$	(step, 2nd clause of <code>power</code>)
$\implies n * \text{power}(n, k)$	(math)
$\implies n * n^k$	(IH)
$\implies n \cdot n^k$	

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$	
$\Rightarrow n * \text{power}(n, k+1-1)$	(step, 2nd clause of <code>power</code>)
$\Rightarrow n * \text{power}(n, k)$	(math)
$\Rightarrow n * n^k$	(IH)
$\Rightarrow n \cdot n^k$	(evaluation rule for <code>*</code>)

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$	
$\implies n * \text{power}(n, k+1-1)$	(step, 2nd clause of <code>power</code>)
$\implies n * \text{power}(n, k)$	(math)
$\implies n * n^k$	(IH)
$\implies n \cdot n^k$	(evaluation rule for <code>*</code>)
$\implies n^{k+1}$	

Let's verify our naive version of power

```
fun power (_:int, 0:int) : int = 1
  | power (n:int, k:int) : int = n * power(n, k-1)
```

Inductive case: Step from k to $k+1$, with $k \geq 0$.

IH: $\text{power}(n, k)$ evaluates to n^k , for $k \geq 0$ and all integers n .

Need to show: $\text{power}(n, k+1)$ evaluates to n^{k+1} .

Showing:

$\text{power}(n, k+1)$	
$\Rightarrow n * \text{power}(n, k+1-1)$	(step, 2nd clause of <code>power</code>)
$\Rightarrow n * \text{power}(n, k)$	(math)
$\Rightarrow n * n^k$	(IH)
$\Rightarrow n \cdot n^k$	(evaluation rule for <code>*</code>)
$\Rightarrow n^{k+1}$	(math)

Let's verify our more efficient version of power, powere

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Let's verify our more efficient version of power, powere

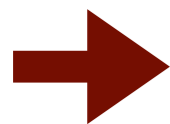
```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

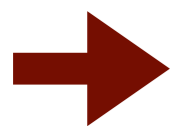


Proof by ???

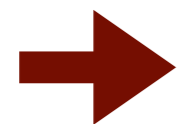
Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



Proof by ???

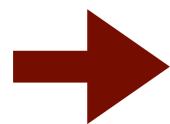


Note: k does no longer decrease by one!

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



Proof by strong induction on k .

Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.

Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.



base case

Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.

Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.



inductive step

Strong induction

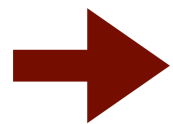
To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.

Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.

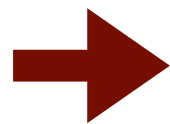


Note: allowed to appeal to IH for any $k' < k$!

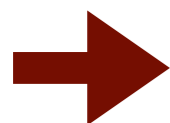
Strong induction

To prove a property $P(n)$ for every natural number n :

- show that $P(0)$ holds
- then, show that for all $k > 0$,
 $P(k)$ follows logically from $\{P(0), \dots, P(k-1)\}$.



Note: allowed to appeal to IH for any $k' < k$!



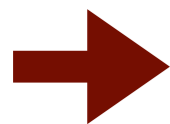
For mathematical induction, IH can only be appealed to for the immediate predecessor!

Let's verify our more efficient version of power, powere

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .



Proof by strong induction on k .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

➔ Proof by strong induction on k .

➔ Notice, the code tells us what induction principle to use!

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
| powere (n:int, k:int) : int =
  if even(k)
  then square(powere(n, k div 2))
  else n * powere(n, k-1)
```

not immediate predecessor!

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

→ Proof by strong induction on k .

→ Notice, the code tells us what induction principle to use!

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
| powere (n:int, k:int) : int =
  if even(k)
  then square(powere(n, k div 2))
  else n * powere(n, k-1)
```

not immediate predecessor!

Theorem: $\text{power}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

- ➔ Proof by strong induction on k .
- ➔ Notice, the code tells us what induction principle to use!
- ➔ Let's do the proof together!

Let's verify our more efficient version of power, powere

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Base case: $k = 0$.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Base case: $k = 0$.

Need to show: $\text{powere}(n, 0)$ evaluates to n^0 , for all n . Note: $n^0 = 1$.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Base case: $k = 0$.

Need to show: $\text{powere}(n, 0)$ evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Base case: $k = 0$.

Need to show: $\text{powere}(n, 0)$ evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

$\text{powere}(n, 0)$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Base case: $k = 0$.

Need to show: $\text{powere}(n, 0)$ evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

$\text{powere}(n, 0)$
 $\Rightarrow 1$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Theorem: $\text{powere}(n, k)$ evaluates to n^k , for all integer values $k \geq 0$ and all integer values n .

Proof: By strong induction on k .

Base case: $k = 0$.

Need to show: $\text{powere}(n, 0)$ evaluates to n^0 , for all n . Note: $n^0 = 1$.

Showing:

$\text{powere}(n, 0)$
 $\implies 1$ (step, 1st clause of `powere`)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

IH: $\text{powere}(n, k')$ evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

IH: $\text{powere}(n, k')$ evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n .

Need to show: $\text{powere}(n, k)$ evaluates to n^k , for all integers n .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

IH: $\text{powere}(n, k')$ evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n .

Need to show: $\text{powere}(n, k)$ evaluates to n^k , for all integers n .

Showing:

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

IH: $\text{powere}(n, k')$ evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n .

Need to show: $\text{powere}(n, k)$ evaluates to n^k , for all integers n .

Showing:

$\text{powere}(n, k)$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

IH: $\text{powere}(n, k')$ evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n .

Need to show: $\text{powere}(n, k)$ evaluates to n^k , for all integers n .

Showing:

$\text{powere}(n, k)$
 \Rightarrow if even(k) (step, 2nd clause of powere)
then square(powere(n, k div 2))
else $n * \text{powere}(n, k-1)$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

IH: $\text{powere}(n, k')$ evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n .

Need to show: $\text{powere}(n, k)$ evaluates to n^k , for all integers n .

Showing:

$\text{powere}(n, k)$
 \implies if even(k) (step, 2nd clause of powere)
then square(powere(n, k div 2))
else $n * \text{powere}(n, k-1)$

Distinguish two subcases, depending on whether k is even or odd.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$
 $\Rightarrow \text{square}(\text{powere}(n, k \text{ div } 2))$ (by assumption about even)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$

$\Rightarrow \text{square}(\text{powere}(n, k \text{ div } 2))$ (by assumption about **even**)

$\Rightarrow \text{square}(\text{powere}(n, k'))$ (since $k = 2k'$, assuming **div** is correct)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$
 $\implies \text{square}(\text{powere}(n, k \text{ div } 2))$ (by assumption about **even**)
 $\implies \text{square}(\text{powere}(n, k'))$ (since $k = 2k'$, assuming **div** is correct)
 $\implies \text{square}(n^{k'})$ (IH)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$
 $\implies \text{square}(\text{powere}(n, k \text{ div } 2))$ (by assumption about even)
 $\implies \text{square}(\text{powere}(n, k'))$ (since $k = 2k'$, assuming div is correct)
 $\implies \text{square}(n^{k'})$ (IH)
 $\implies (n^{k'})^2$ (by Lemma)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$
 $\Rightarrow \text{square}(\text{powere}(n, k \text{ div } 2))$ (by assumption about even)
 $\Rightarrow \text{square}(\text{powere}(n, k'))$ (since $k = 2k'$, assuming div is correct)
 $\Rightarrow \text{square}(n^{k'})$ (IH)
 $\Rightarrow (n^{k'})^2$ (by Lemma)

Lemma: For every integer value n , $\text{square}(n)$ evaluates to n^2 .

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$
 $\implies \text{square}(\text{powere}(n, k \text{ div } 2))$ (by assumption about even)
 $\implies \text{square}(\text{powere}(n, k'))$ (since $k = 2k'$, assuming div is correct)
 $\implies \text{square}(n^{k'})$ (IH)
 $\implies (n^{k'})^2$ (by Lemma)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

$$\begin{aligned} & \text{powere}(n, k) \\ \implies & \text{square}(\text{powere}(n, k \text{ div } 2)) \quad (\text{by assumption about even}) \\ \implies & \text{square}(\text{powere}(n, k')) \quad (\text{since } k = 2k', \text{ assuming div is correct}) \\ \implies & \text{square}(n^{k'}) \quad (\text{IH}) \\ \implies & (n^{k'})^2 \quad (\text{by Lemma}) \\ = & n^{2k'} = n^k \quad (\text{math}) \end{aligned}$$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k' + 1$, for some $k' < k$, assuming correctness of even.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k' + 1$, for some $k' < k$, assuming correctness of even.

Showing:

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k' + 1$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k' + 1$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$

$\Rightarrow n * (\text{powere}(n, k-1))$ (by assumption about even)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k' + 1$, for some $k' < k$, assuming correctness of even.

Showing:

$\text{powere}(n, k)$

$\Rightarrow n * (\text{powere}(n, k-1))$ (by assumption about even)

$\Rightarrow n * n^{k-1}$ (IH)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
    if even(k)
    then square(powere(n, k div 2))
    else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k' + 1$, for some $k' < k$, assuming correctness of even.

Showing:

$$\begin{aligned} & \text{powere}(n, k) \\ \Rightarrow & n * (\text{powere}(n, k-1)) && \text{(by assumption about even)} \\ \Rightarrow & n * n^{k-1} && \text{(IH)} \\ \Rightarrow & n^k && \text{(math)} \end{aligned}$$

That's all for today. See you on Thursday!