Recursion and Induction

15-150

Lecture 3: September 1, 2026

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Carnegie Mellon University

Recap of week 1

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Functional programming

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→ is concerned with **evaluation** of expressions (no mutation!)

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Types, expressions, values

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Types, expressions, values

- **types** as specifications

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Types, expressions, values

- **types** as specifications
- observation: once your program type checks, it works!

Recap of week 1

Extensional equivalence (≅)

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→ “Two things are equal if the behave the same”

Recap of week 1

we'll revisit exact definition

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- **shadowing** of bindings

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- **function** declarations bind a lambda to the function identifier

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Declarations, binding and scope

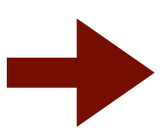
- **shadowing** of bindings
- **function** declarations bind a lambda to the function identifier
- values of any bindings existing at declaration time are substituted for free variables in lambda expression

Recap of week 1

Pattern matching

Recap of week 1

Pattern matching



patterns are used at binding sites of values

Recap of week 1

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  - eg, val bindings, function arguments, case expressions
- allow us to **match** against an expected value
- allow us to **decompose** a value in its constituent parts, introducing appropriate **bindings** for parts

Recap of week 1

5-step methodology of function declarations

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1. function name and type

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Today, we add a 6th step:

Recap of week 1

5-step methodology of function declarations

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Today, we add a 6th step:

6. correctness proof

Today's topic: functional correctness

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Let's prove our programs correct, one function at a time!

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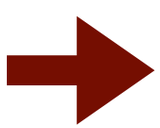

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- we will use three kinds of **induction**:
  - **mathematical** induction
  - **strong** induction
  - **structural** induction
- we consider how expressions are evaluated
- we may appeal to mathematical properties and assume that SML implements them correctly

An example: compute n^k

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(* power : (int * int) -> int
   REQUIRES: k >= 0
   ENSURES: power(n,k) ==> n^k, with 0^0 = 1.
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recursive call

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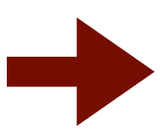

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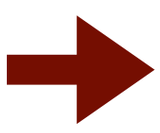

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exponent k is even

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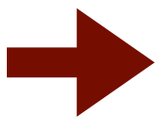

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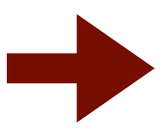

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- Let’s use mathematical induction!

Mathematical (simple, weak) induction

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To prove a property P(n) for every natural number n:

- show that P(0) holds
- then, show that for all $k \geq 0$,
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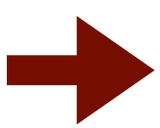

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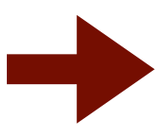

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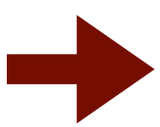

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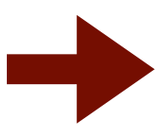

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Theorem: `power(n, k)` evaluates to n^k, for all integer values $k \geq 0$ and all integer values `n`.

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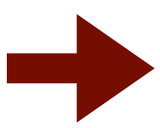

Proof by mathematical induction on ???

Let's verify our naive version of power

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fun power (_:int, 0:int) : int = 1
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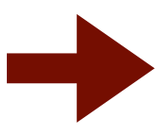

Proof by mathematical induction on k.

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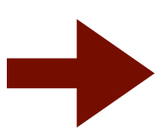

Proof by mathematical induction on k.

k is the integer that gets smaller!

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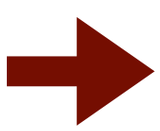

Proof by mathematical induction on k.

k is the integer that gets smaller!

needed for applying IH!

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fun power (_:int, 0:int) : int = 1
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- Proof by mathematical induction on k.
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Proof: By mathematical induction on k.

Base case: k = 0.

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Proof: By mathematical induction on k.

Base case: k = 0.

Need to show: `power(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

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Showing:

```
power(n,0)
```

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Showing:

```
    power(n,0)
⟹  1
```

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Proof: By mathematical induction on k.

Base case: k = 0.

Need to show: `power(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

Showing:

`power(n,0)`

$\Longrightarrow$ `1` (step, 1st clause of `power`)

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Inductive case: Step from k to k+1, with k $\geq$ 0.

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Showing:

```
    power(n,k+1)
⟹  n * power(n,k+1-1)
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Showing:

```
    power(n,k+1)
⟹  n * power(n,k+1-1)          (step, 2nd clause of power)
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Showing:

```
    power(n,k+1)
⟹  n * power(n,k+1-1)        (step, 2nd clause of power)
⟹  n * power(n,k)            (math)
```

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	`power(n,k+1)`	
$\Longrightarrow$	`n * power(n,k+1-1)`	(step, 2nd clause of `power`)
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$\Longrightarrow$	`n *` n^k	

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Showing:

`power(n,k+1)`	
$\Longrightarrow$ `n * power(n,k+1-1)`	(step, 2nd clause of `power`)
$\Longrightarrow$ `n * power(n,k)`	(math)
$\Longrightarrow$ `n * ` n^k	(IH)
$\Longrightarrow$ `n ·` n^k	(evaluation rule for $*$)

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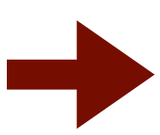

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→ Proof by ???

→ Note: k does no longer decrease by one!

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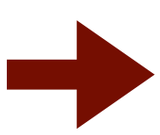

Proof by strong induction on k.

Strong induction

To prove a property P(n) for every natural number n:

- show that P(0) holds
- then, show that for all k > 0,
  P(k) follows logically from {P(0), …, P(k-1)}.

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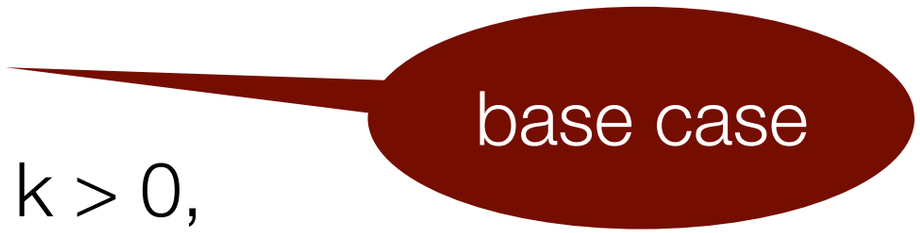

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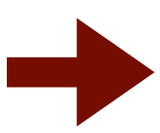

Note: allowed to appeal to IH for any k’ < k!

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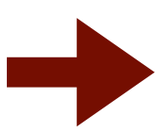

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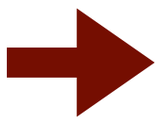

For mathematical induction, IH can only be appealed to for the immediate predecessor!

Let’s verify our more efficient version of power, powere

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- Notice, the code tells us what induction principle to use!

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- Let's do the proof together!

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Proof: By strong induction on k.

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Theorem: `powere(n, k)` evaluates to n^k, for all integer values $k \geq 0$ and all integer values n.

Proof: By strong induction on k.

Base case: k = 0.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Theorem: `powere(n, k)` evaluates to n^k, for all integer values `k` $\geq$ `0` and all integer values `n`.

Proof: By strong induction on k.

Base case: k = 0.

Need to show: `powere(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

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```
fun powere (_:int, 0:int) : int = 1
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Proof: By strong induction on k.

Base case: k = 0.

Need to show: `powere(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

Showing:

Let's verify our more efficient version of power, powere

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fun powere (_:int, 0:int) : int = 1
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Base case: k = 0.

Need to show: `powere(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

Showing:

```
powere(n,0)
```

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
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Base case: k = 0.

Need to show: `powere(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

Showing:

```
    powere(n,0)
```

$\Longrightarrow$ `1`

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
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Theorem: `powere(n, k)` evaluates to n^k, for all integer values $k \geq 0$ and all integer values `n`.

Proof: By strong induction on k.

Base case: k = 0.

Need to show: `powere(n,0)` evaluates to n^0, for all n. Note: $n^0 = 1$.

Showing:

`powere(n,0)`

$\Longrightarrow$ `1` (step, 1st clause of `powere`)

Let's verify our more efficient version of power, powere

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fun powere (_:int, 0:int) : int = 1
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```

Inductive case: k > 0.

Let's verify our more efficient version of power, powere

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fun powere (_:int, 0:int) : int = 1
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Inductive case: k > 0.

IH: `powere(n,k')` evaluates to $n^{k'}$, for $0 \leq k' < k$ and all integers n.

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Showing:

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Showing:

`powere(n,k)`

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Need to show: `powere(n,k)` evaluates to n^k, for all integers n.

Showing:

```
    powere(n,k)
⟹  if even(k)                          (step, 2nd clause of powere)
    then square(powere(n,k div 2))
    else n * powere(n,k-1)
```

Let's verify our more efficient version of power, powere

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fun powere (_:int, 0:int) : int = 1
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Showing:

```
    powere(n,k)
⟹  if even(k)                          (step, 2nd clause of powere)
    then square(powere(n,k div 2))
    else n * powere(n,k-1)
```

Distinguish two subcases, depending on whether k is even or odd.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
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      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `square(powere(n, k div 2))` (by assumption about `even`)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `square(powere(n, k div 2))` (by assumption about **`even`**)

$\Longrightarrow$ `square(powere(n, k'))` (since k = 2k', assuming **`div`** is correct)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `square(powere(n, k div 2))` (by assumption about **`even`**)

$\Longrightarrow$ `square(powere(n, k'))` (since k = 2k', assuming **`div`** is correct)

$\Longrightarrow$ `square(`$n^{k'}$`)` (IH)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `square(powere(n, k div 2))` (by assumption about `even`)

$\Longrightarrow$ `square(powere(n, k'))` (since $k = 2k'$, assuming `div` is correct)

$\Longrightarrow$ `square(`$n^{k'}$`)` (IH)

$\Longrightarrow$ $(n^{k'})^2$ (by Lemma)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `square(powere(n, k div 2))` (by assumption about **`even`**)

$\Longrightarrow$ `square(powere(n, k'))` (since k = 2k', assuming **`div`** is correct)

$\Longrightarrow$ `square(`$n^{k'}$`)` (IH)

$\Longrightarrow$ $(n^{k'})^2$ (by Lemma)

Lemma: For every integer value n, square(n) evaluates to n^2.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k', for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `square(powere(n, k div 2))` (by assumption about **`even`**)

$\Longrightarrow$ `square(powere(n, k'))` (since k = 2k', assuming **`div`** is correct)

$\Longrightarrow$ `square(`$n^{k'}$`)` (IH)

$\Longrightarrow$ $(n^{k'})^2$ (by Lemma)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: $k > 0$.

Case: $k = 2k'$, for some $k' < k$, assuming correctness of even.

Showing:

	`powere(n,k)`	
$\Longrightarrow$	`square(powere(n, k div 2))`	(by assumption about `even`)
$\Longrightarrow$	`square(powere(n, k'))`	(since $k = 2k'$, assuming `div` is correct)
$\Longrightarrow$	`square(`$n^{k'}$`)`	(IH)
$\Longrightarrow$	$(n^{k'})^2$	(by Lemma)
=	$n^{2k'} = n^k$	(math)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k'+1, for some k' < k, assuming correctness of even.

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
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      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k'+1, for some k' < k, assuming correctness of even.

Showing:

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k'+1, for some k' < k, assuming correctness of even.

Showing:

```
powere(n,k)
```

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k'+1, for some k' < k, assuming correctness of even.

Showing:

```
    powere(n,k)
```

$\Longrightarrow$ `n * (powere(n, k-1))` (by assumption about `even`)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k'+1, for some k' < k, assuming correctness of even.

Showing:

`powere(n,k)`

$\Longrightarrow$ `n * (powere(n, k-1))` (by assumption about `even`)

$\Longrightarrow$ `n *` n^{k-1} (IH)

Let's verify our more efficient version of power, powere

```
fun powere (_:int, 0:int) : int = 1
  | powere (n:int, k:int) : int =
      if even(k)
      then square(powere(n, k div 2))
      else n * powere(n, k-1)
```

Inductive case: k > 0.

Case: k = 2k'+1, for some k' < k, assuming correctness of even.

Showing:

	`powere(n,k)`	
$\Longrightarrow$	`n * (powere(n, k-1))`	(by assumption about `even`)
$\Longrightarrow$	`n` * n^{k-1}	(IH)
$\Longrightarrow$	n^k	(math)

That's all for today. See you on Thursday!