Variational (weak) form of linear elasticity

In this handout I derive the weak form of the equations of linear elasticity in symbolic form. This requires some facility with tensors. First, some notation: For vector $\mathbf{v}$ and tensors $\mathbf{A}$ and $\mathbf{B}$,

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}, \quad (\nabla \cdot \mathbf{A})_i = \sum_j \frac{\partial A_{ij}}{\partial x_j},$$

$$(\mathbf{A} \mathbf{v})_i = \sum_j A_{ij} v_j, \quad \mathbf{A} \cdot \mathbf{B} = \sum_i \sum_j A_{ij} B_{ij}.$$  

The linear elastic deformation of an isotropic solid is described in terms of the stress tensor $\sigma$, the strain tensor $\varepsilon$, the displacement vector $\mathbf{u}$, the traction vector $\mathbf{t}$, the body force vector $\mathbf{f}$, and the Lamé constants $\mu$ and $\lambda$. The governing field equations consist of the equilibrium equation

$$-\nabla \cdot \sigma = \mathbf{f},$$

the strain-displacement relation

$$\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T),$$

and the constitutive law

$$\sigma = 2\mu \varepsilon + \lambda (\nabla \cdot \mathbf{u}) \mathbf{I}.$$  

The displacement form of the equilibrium equation is thus

$$-\nabla \cdot [\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda (\nabla \cdot \mathbf{u}) \mathbf{I}] = \mathbf{f}.$$  

Finally, we need some boundary conditions, for example

$$\mathbf{u} = \hat{\mathbf{u}} \text{ on } \Gamma_D,$$

$$\mathbf{t} \equiv \sigma \mathbf{n} = \hat{\mathbf{t}} \text{ on } \Gamma_N.$$

Now we’re ready to construct the weak form. We multiply the residual by a test (vector) function $\mathbf{v}$ and integrate over the domain $\Omega$,

$$\int_{\Omega} \mathbf{v} \cdot (-\nabla \cdot \sigma - \mathbf{f}) \, d\Omega = 0.$$  

Using a Green’s formula, we obtain

$$\int_{\Omega} \nabla \mathbf{v} \cdot \sigma \, d\Omega = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_{\Gamma} \mathbf{v} \cdot \sigma \mathbf{n} \, d\Gamma.$$  

Then we substitute for the stress and traction

$$\int_{\Omega} \nabla \mathbf{v} \cdot [\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda (\nabla \cdot \mathbf{u}) \mathbf{I}] \, d\Omega = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_{\Gamma} \mathbf{v} \cdot \mathbf{t} \, d\Gamma.$$  

Finally, define $\mathcal{U}$ as the space of all vector functions whose derivatives are square integrable and that satisfy the essential boundary condition, and $\mathcal{V}$ as the space of all vector functions whose derivatives are square integrable and that vanish on $\Gamma_D$. Rearranging and making use of the boundary conditions, we obtain the weak form of the linear elasticity problem: Find $\mathbf{u} \in \mathcal{U}$ such that

$$\int_{\Omega} \frac{\mu}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \, d\Omega + \int_{\Omega} \lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) \, d\Omega = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_{\Gamma_N} \mathbf{v} \cdot \hat{\mathbf{t}} \, d\Gamma$$

for all $\mathbf{v} \in \mathcal{V}$. This form is clearly symmetric.