The Perceptron Algorithm, Margins

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The Perceptron Algorithm

Simple learning algorithm for supervised classification analyzed via geometric margins in the 50's [Rosenblatt'57] .

Originally introduced in the online learning scenario.

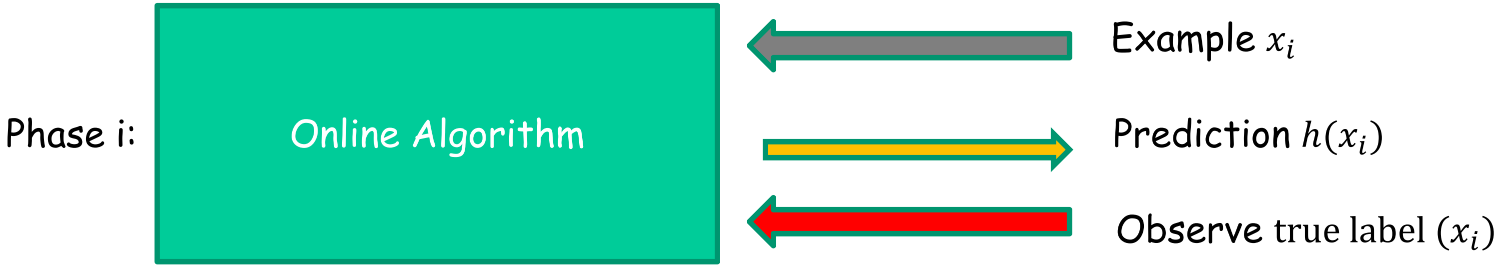
- Online Learning Model
- Perceptron Algorithm
- Its Guarantees under large margins

The Online Learning Model

- Example arrive sequentially.
- We need to make a prediction.

  Afterwards observe the outcome.

For i=1, 2, …, :



Phase i:

Online Algorithm

Example x_i

Prediction $h(x_i)$

Observe true label (x_i)

Mistake bound model

- **Goal**: Minimize the number of mistakes.
- Analysis wise, make no distributional assumptions, consider a worst sequence of examples.

The Online Learning Model. Motivation

- Email classification (distribution of both spam and regular mail changes over time, but the target function stays fixed - last year's spam still looks like spam).

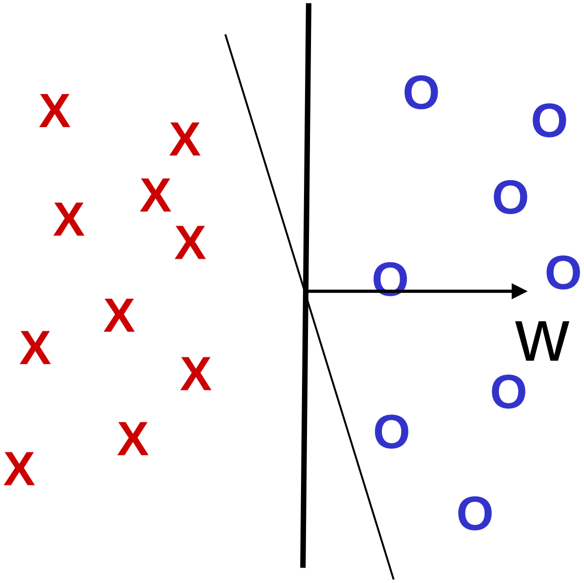
- Recommendation systems. Recommending movies, etc.

- Predicting whether a user will be interested in a new news article or not.

- Add placement in a new market.

Linear Separators

- Feature space $\mathrm{X} = \mathrm{R}^{\mathrm{d}}$
- Hypothesis class of linear decision surfaces in R^d.
  - $\mathrm{h(x) = w \cdot\ x\ + w_0}$, if $h(x) \geq\ 0$, then label **x** as **+**, otherwise label it as **-**



Trick: Without loss of generality $\mathrm{w_0 = 0}$.

Proof: Can simulate a non-zero threshold with a dummy input feature x_0 that is always set up to 1.

- $x = (x_1, \dots, x_d) \rightarrow \tilde{x} = (x_1, \dots, x_d, 1)$
- $\mathrm{w \cdot\ x\ + w_0} \geq 0$ iff $(w_1, \dots, w_d, \mathrm{w_0}\ ) \cdot \tilde{x} \geq 0$

where $\mathrm{w} = (w_1, \dots, w_d)$

Linear Separators: Perceptron Algorithm

- Set t=1, start with the all zero vector w_1.
- Given example x, predict positive iff $w_t \cdot x \geq 0$
- On a mistake, update as follows:
  - Mistake on positive, then update $w_{t+1} \leftarrow w_t + x$
  - Mistake on negative, then update $w_{t+1} \leftarrow w_t - x$

Natural greedy procedure:

- If true label of x is +1 and w_t incorrect on x we have $w_t \cdot x < 0$, $w_{t+1} \cdot x \leftarrow w_t \cdot x + x \cdot x = w_t \cdot x + ||x||^2$, so more chance w_{t+1} classifies x correctly.
  - Similarly for mistakes on negative examples.

Linear Separators: Perceptron Algorithm

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Note: w_t is weighted sum of incorrectly classified examples

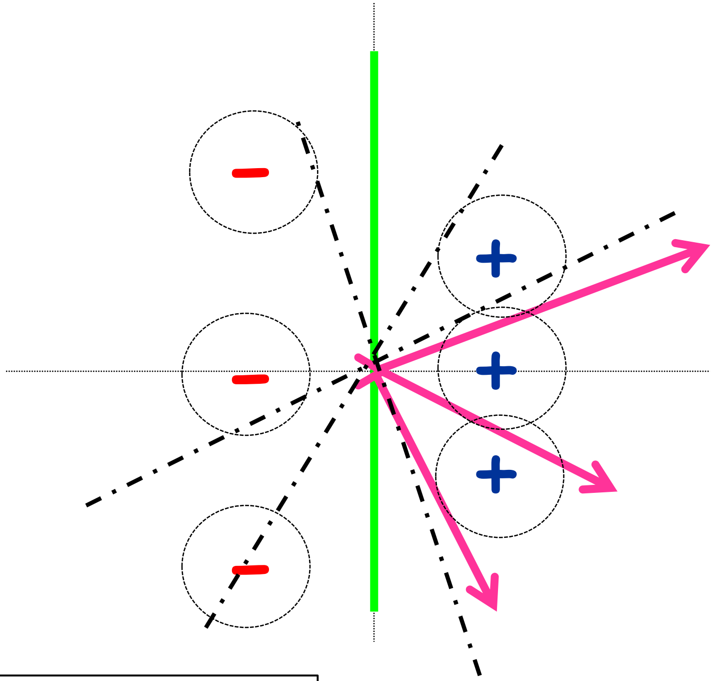
$$w_t = a_{i_1} x_{i_1} + \cdots + a_{i_k} x_{i_k}$$

$$w_t \cdot x = a_{i_1} x_{i_1} \cdot x + \cdots + a_{i_k} x_{i_k} \cdot x$$

Important when we talk about kernels.

Perceptron Algorithm: Example

Example:	$(-1,2)$ $-$	✗
	$(1,0)$ $+$	✓
	$(1,1)$ $+$	✗
	$(-1,0)$ $-$	✓
	$(-1,-2)$ $-$	✗
	$(1,-1)$ $+$	✓



$w_1 = (0,0)$

$w_2 = w_1 - (-1,2) = (1,-2)$

$w_3 = w_2 + (1,1) = (2,-1)$

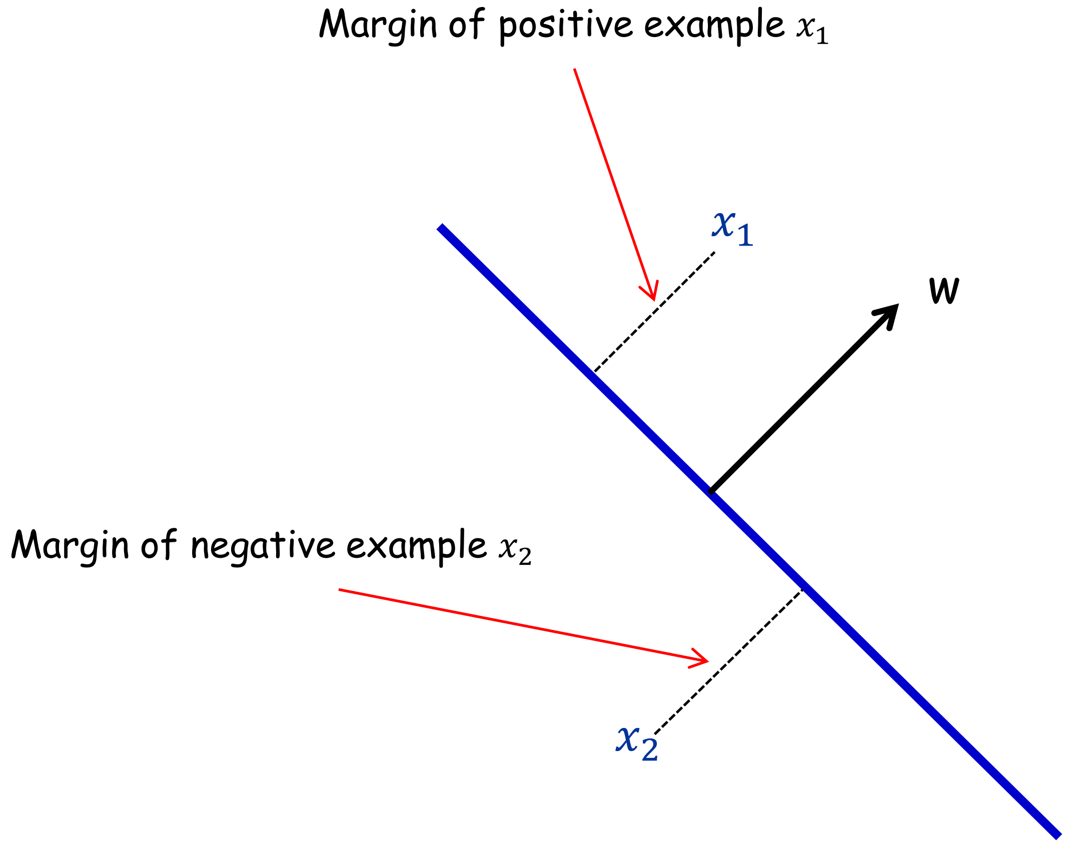
$w_4 = w_3 - (-1,-2) = (3,1)$

Algorithm:

- Set t=1, start with all-zeroes weight vector w_1.
- Given example x, predict positive iff $w_t \cdot x \geq 0$.
  - On a mistake, update as follows:
    - Mistake on positive, update $w_{t+1} \leftarrow w_t + x$
    - Mistake on negative, update $w_{t+1} \leftarrow w_t - x$

Geometric Margin

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$



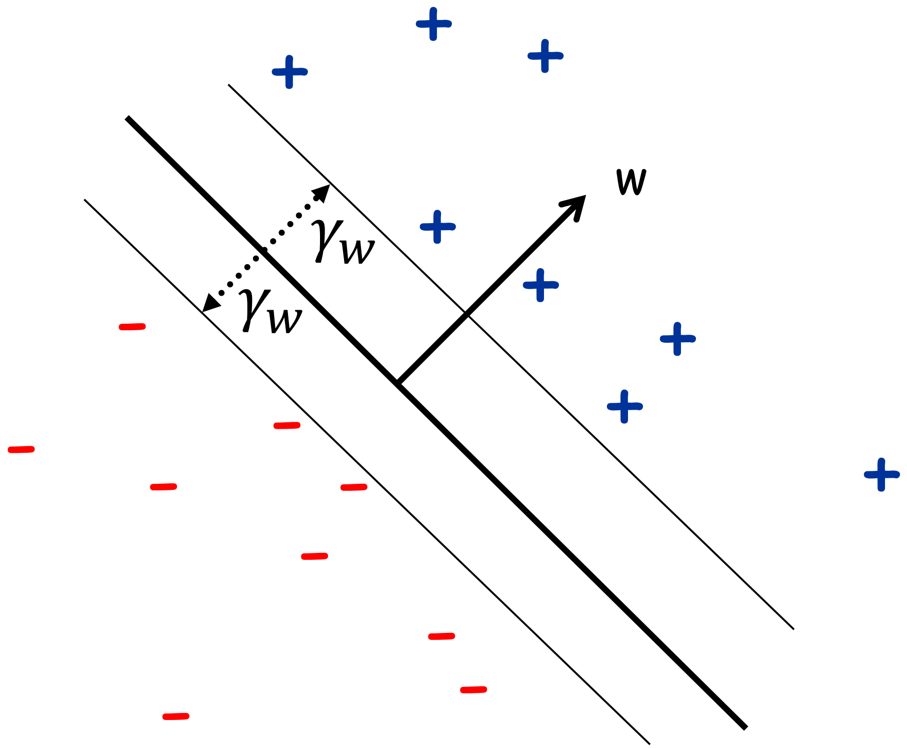
Margin of positive example x_1

Margin of negative example x_2

Geometric Margin

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$

Definition: The margin γ_w of a set of examples S wrt a linear separator w is the smallest margin over points $x \in S$.



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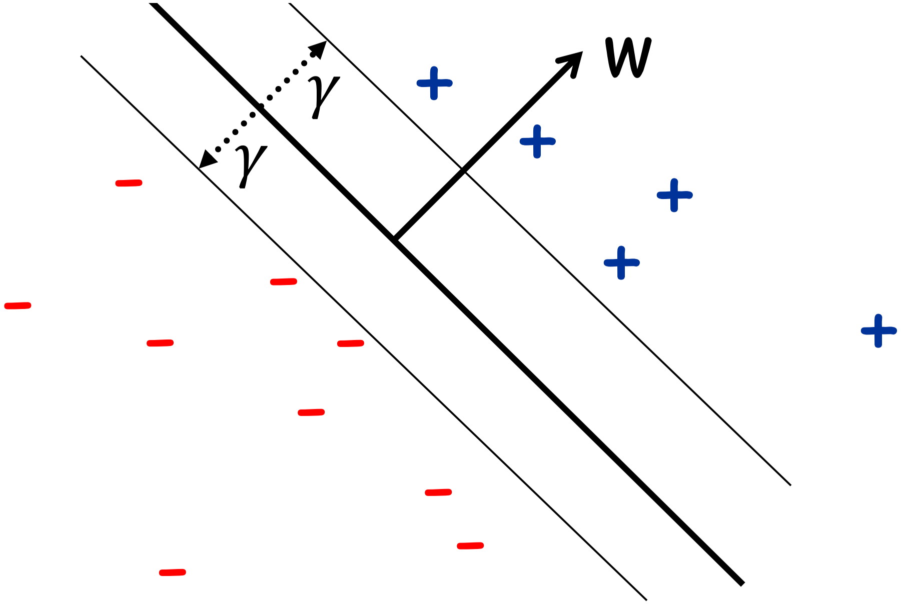
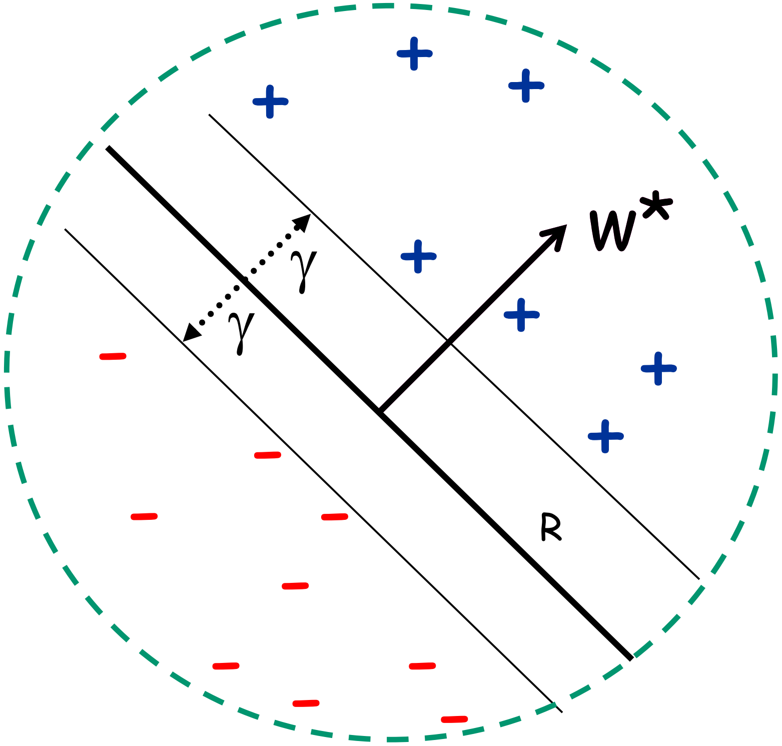
Definition: The margin γ of a set of examples S is the maximum γ_w over all linear separators w.



Perceptron: Mistake Bound

Guarantee: If data has margin γ and all points inside a ball of radius R, then Perceptron makes $\leq (R/\gamma)^2$ mistakes.

(Normalized margin: multiplying all points by 100, or dividing all points by 100, doesn't change the number of mistakes; algo is invariant to scaling.)



Perceptron Algorithm: Analysis

Guarantee: If data has margin γ and all points inside a ball of radius R, then Perceptron makes $\leq (R/\gamma)^2$ mistakes.

Update rule:

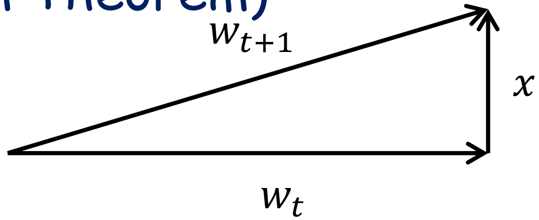
- Mistake on positive: $w_{t+1} \leftarrow w_t + x$
- Mistake on negative: $w_{t+1} \leftarrow w_t - x$

Proof:

Idea: analyze $w_t \cdot w^*$ and $\|w_t\|$, where w^* is the max-margin sep, $\|w^*\| = 1$.

Claim 1: $w_{t+1} \cdot w^* \geq w_t \cdot w^* + \gamma$. (because $l(x)x \cdot w^* \geq \gamma$)

Claim 2: $\|w_{t+1}\|^2 \leq \|w_t\|^2 + R^2$. (by Pythagorean Theorem)

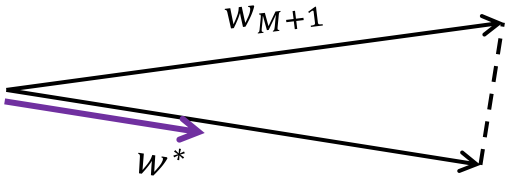


After M mistakes:

$w_{M+1} \cdot w^* \geq \gamma M$ (by Claim 1)

$\|w_{M+1}\| \leq R\sqrt{M}$ (by Claim 2)

$w_{M+1} \cdot w^* \leq \|w_{M+1}\|$ (since w^* is unit length)



So, $\gamma M \leq R\sqrt{M}$, so $M \leq \left(\frac{R}{\gamma}\right)^2$.

Finding a Consistent Separator

Can use Perceptron in a batch setting too, to find a consistent linear separator given a set S of labeled examples that is linearly separable by margin γ.

- We repeatedly feed the whole set S of labeled examples into the Perceptron algorithm up to $\left(\frac{R}{\gamma}\right)^2 + 1$ rounds, until we get to a point where the current hypothesis is consistent/correct with the whole set S. Note we are guaranteed to reach such a point.
- The runnning time is then polynomial in $\frac{R}{\gamma^2}$ and |S|.

What if there is no perfect separator?

Perceptron can be adapted to the case where there is no perfect separator as long as the so called hinge loss (i.e., the total distance needed to move the points to classify them correctly large margin) is small.

Let TD_γ = total distance needed to move the points to classify them correctly large margin.

After M mistakes:

$\mathrm{w}_{\mathrm{M}+1} \cdot \mathrm{w}^* \geq \gamma \mathrm{M} - \mathrm{TD}_\gamma$ (new Claim 1)

$\|w_{M+1}\| \leq R\sqrt{M}$ (by Claim 2)

So, $\gamma \mathrm{M} - \mathrm{TD}_\gamma \leq \mathrm{R}\sqrt{\mathrm{M}}$, so $\mathrm{M} \leq \left(\frac{\mathrm{R}}{\gamma}\right)^2 + \frac{2}{\gamma}\mathrm{TD}_\gamma$.

Margin Perceptron for Approx Maximizing the Margins

- Set t=1, $w_1 = l(\mathrm{x})\mathrm{x}$, where x is the first example.
- Given example x, predict positive if $w_t \cdot x \geq \frac{\gamma}{2}$

  predict negative if $w_t \cdot x \leq \frac{\gamma}{2}$

  margin mistake if $w_t \cdot x \in (-\frac{\gamma}{2}, \frac{\gamma}{2})$
- On a mistake, update: $w_{t+1} \leftarrow w_t + l(x)x$

Guarantee: If data has margin γ and all points inside a ball of radius R, then Margin Perceptron makes $\leq 8\left(\frac{R}{\gamma}\right)^2 + 4\frac{R}{\gamma}$ mistakes.

Perceptron Discussion

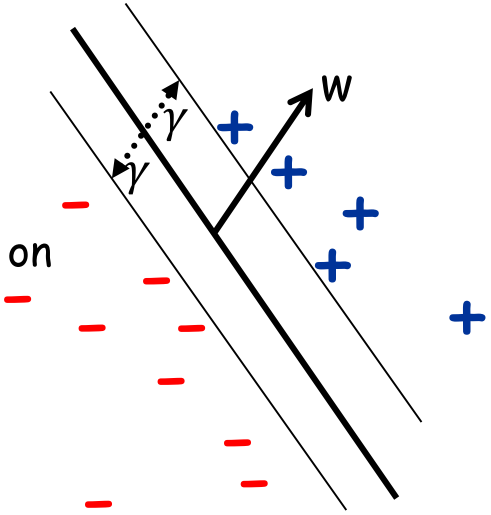
- Simple, but very useful in applications like branch prediction; it also has interesting extensions to structured prediction.

- Can be kernelized to handle non-linear decision boundaries!
  - See future lecture!!!

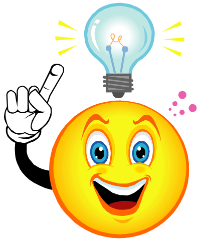
Margin Important Theme in ML

- If large margin, # mistakes Peceptron makes is small (independent on the dim of the ambient space)!

- Large margin can help prevent overfitting.
  - If large margin γ and if alg. produces a large margin classifier, then amount of data needed depends only on R/γ [Bartlett & Shawe-Taylor '99].



- Why not directly search for a large margin classifier?

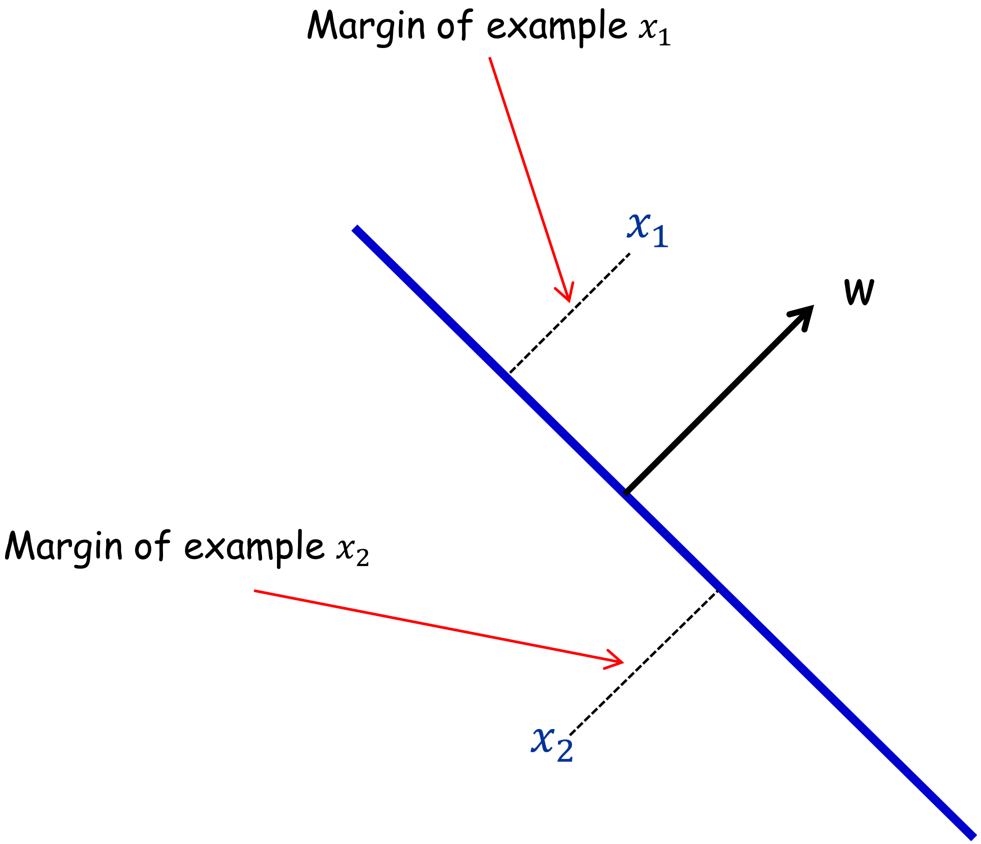


Support Vector Machines (SVMs).

Geometric Margin

WLOG homogeneous linear separators $[w_0 = 0]$.

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$.



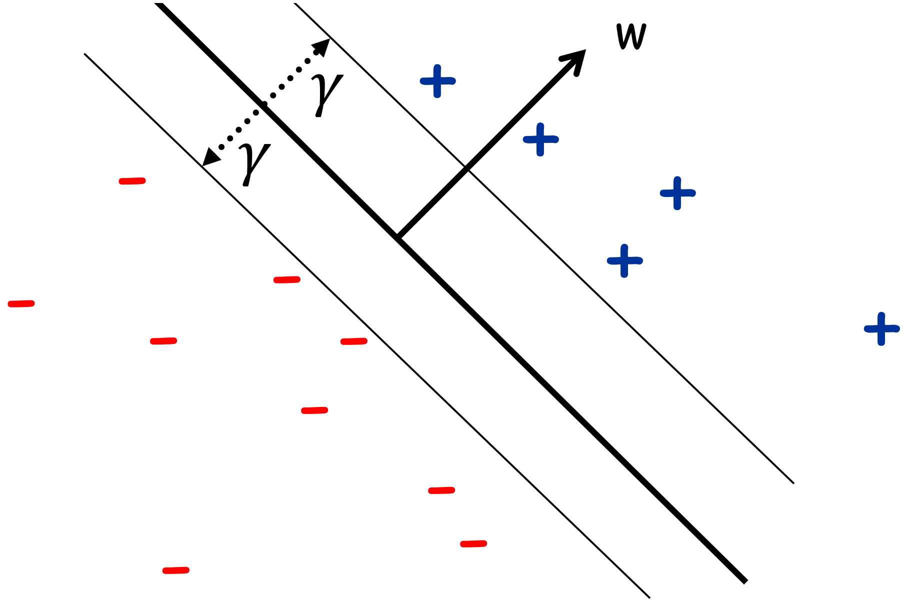
If $||w|| = 1$, margin of x w.r.t. w is $|x \cdot w|$.

Geometric Margin

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$.

Definition: The margin γ_w of a set of examples S wrt a linear separator w is the smallest margin over points $x \in S$.

Definition: The margin γ of a set of examples S is the maximum γ_w over all linear separators w.



Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

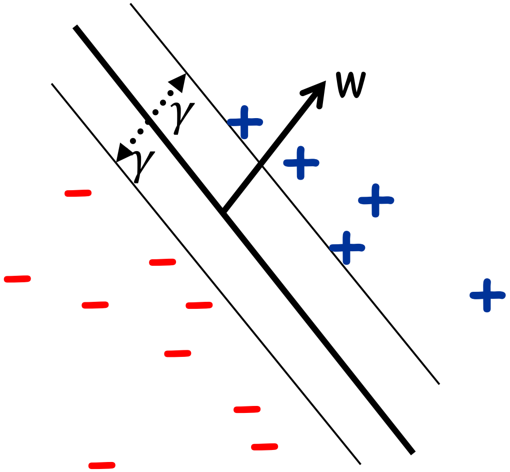
First, assume we know a lower bound on the margin γ

Input: γ, S={(x_1, y_1), …,(x_m, y_m)};

Find: some w where:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

Output: w, a separator of margin γ over S

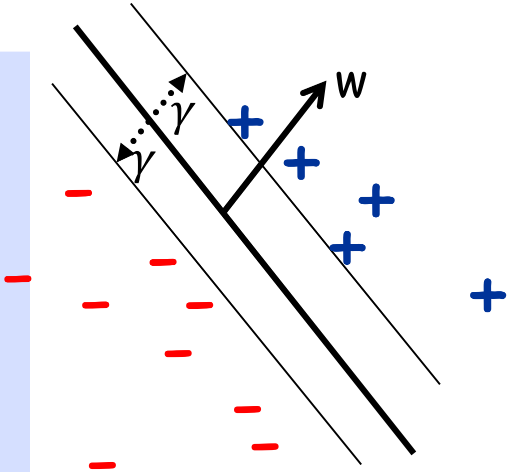


The case where the data is truly linearly separable by margin γ

Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

E.g., search for the best possible γ



Input: S={(x_1, y_1), …,(x_m, y_m)};

Find: some w and maximum γ where:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

Output: maximum margin separator over S

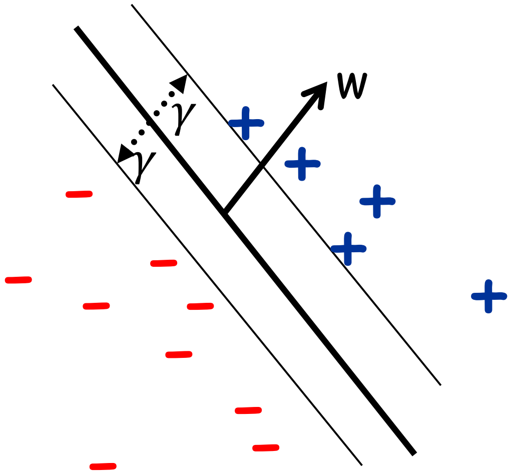
Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

Input: S={(x_1, y_1), …,$(\mathrm{x}_\mathrm{m}, y_\mathrm{m})$};

Maximize γ under the constraint:

- $||\mathrm{w}||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$



Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

Input: S={(x_1, y_1), …,(x_m, y_m)};

Maximize γ under the constraint:

- $||w||^2 = 1$
- For all i, $\underbrace{y_i w \cdot x_i \geq \gamma}_{\text{constraints}}$

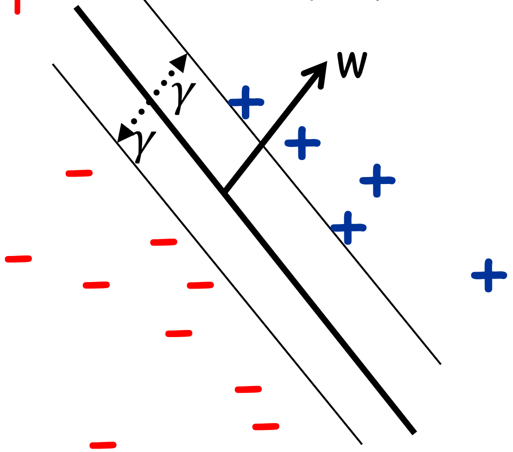
objective function

This is a **constrained optimization** problem.

- Famous example of constrained optimization: **linear programming**, where objective fn is linear, constraints are linear (in)equalities

Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs



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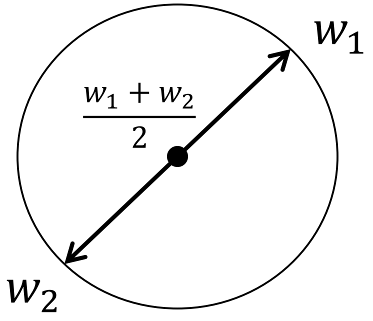
Maximize γ under the constraint:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

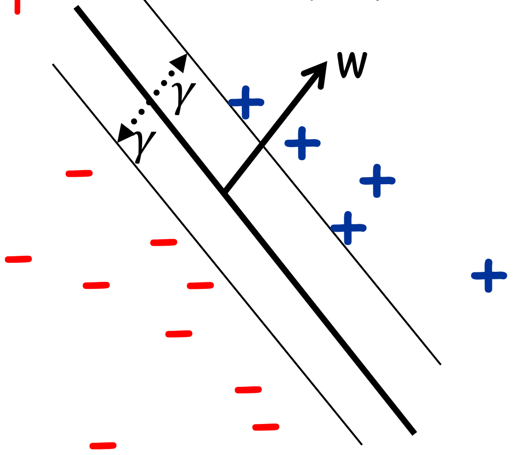
This constraint is non-linear.

In fact, it's even non-convex



Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

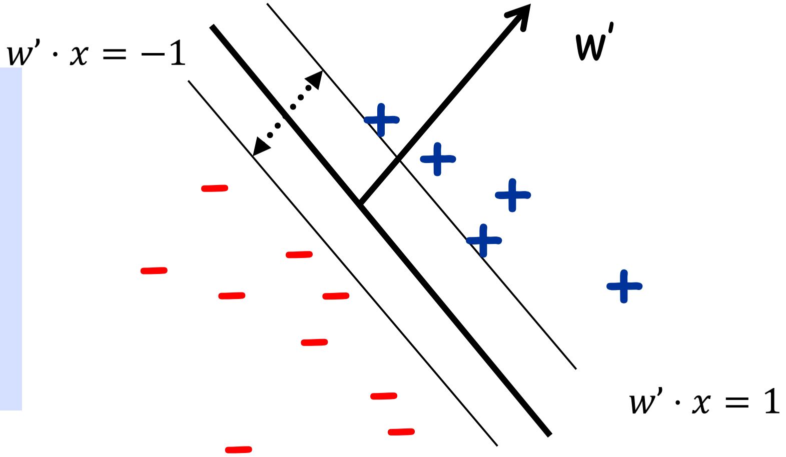


Input: S={(x_1, y_1), …,(x_m, y_m)};

Maximize γ under the constraint:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

$w' = w/\gamma$, then max γ is equiv. to minimizing $||w'||^2$ (since $||w'||^2 = 1/\gamma^2$). So, dividing both sides by γ and writing in terms of w′ we get:



Input: S={(x_1, y_1), …,(x_m, y_m)};

Minimize $||w'||^2$ under the constraint:

- For all i, $y_i w' \cdot x_i \geq 1$

Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

<u>Input</u>: S={$(\mathrm{x}_1, y_1), \ldots, (\mathrm{x}_\mathrm{m}, y_\mathrm{m})$};

$\mathrm{argmin}_w \, ||w||^2$ s.t.:

- For all i, $y_i w \cdot x_i \geq 1$

This is a **constrained optimization** problem.

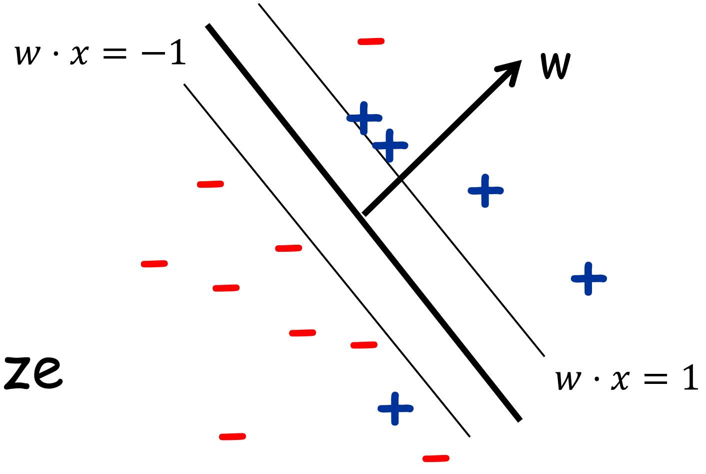
- The objective is convex (quadratic)
- All constraints are linear
- Can solve efficiently (in poly time) using standard **quadratic programing** (QP) software

Support Vector Machines (SVMs)

Question: what if data isn't perfectly linearly separable?

Issue 1: now have two objectives

- maximize margin
- minimize # of misclassifications.



Ans 1: Let's optimize their sum: minimize

$$||w||^2 + C(\text{\# misclassifications})$$

where C is some tradeoff constant.

Issue 2: This is computationally very hard (NP-hard).

[even if didn't care about margin and minimized # mistakes]

Support Vector Machines (SVMs)

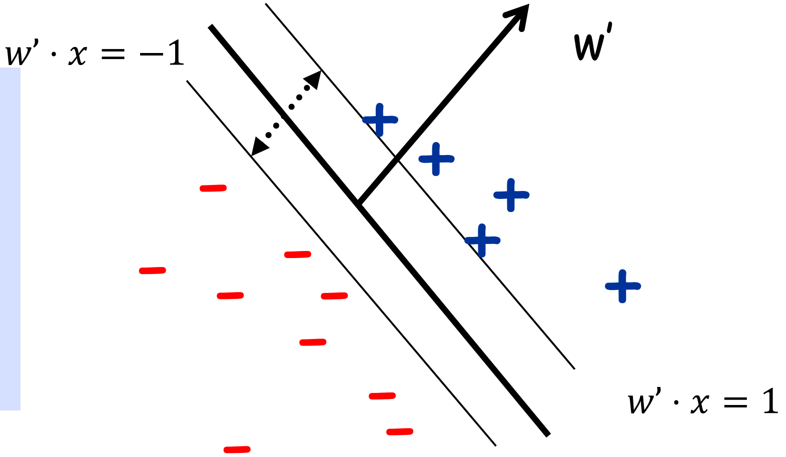
Question: what if data isn't perfectly linearly separable?

Replace "# mistakes" with upper bound called "hinge loss"

Input: S={(x_1, y_1), …,(x_m, y_m)};

Minimize $||w'||^2$ under the constraint:

- For all i, $y_i w' \cdot x_i \geq 1$



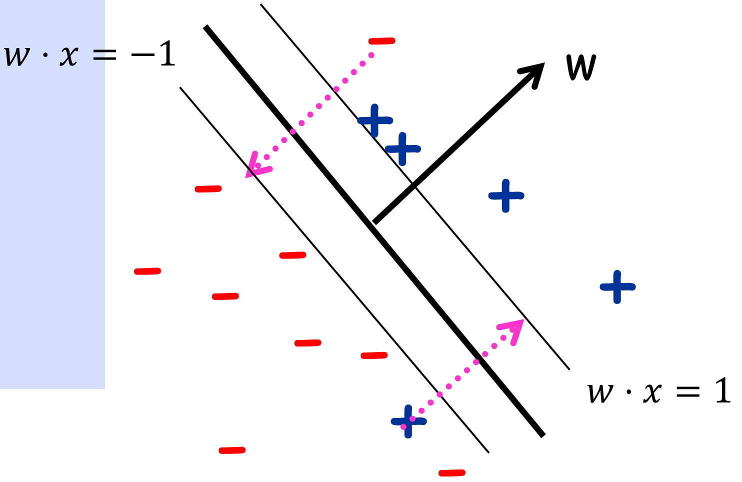
Input: S={(x_1, y_1), …,(x_m, y_m)};

Find $\text{argmin}_{w,\xi_1,\ldots,\xi_m} ||w||^2 + C \sum_i \xi_i$ s.t.:

- For all i, $y_i w \cdot x_i \geq 1 - \xi_i$

  $\xi_i \geq 0$

ξ_i are "slack variables"



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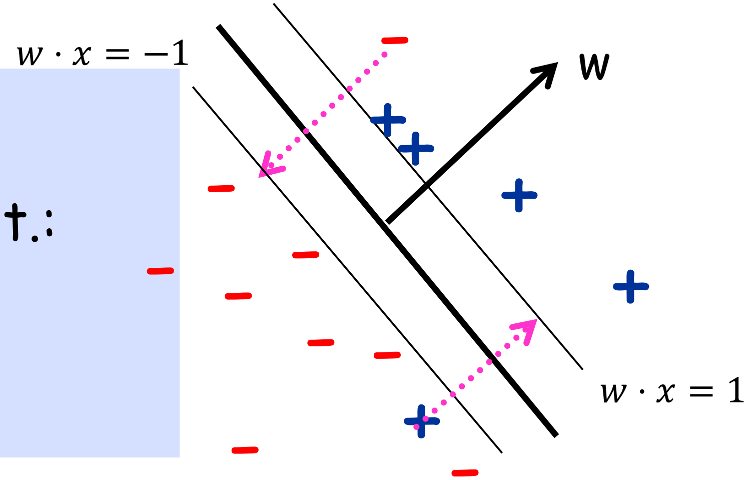
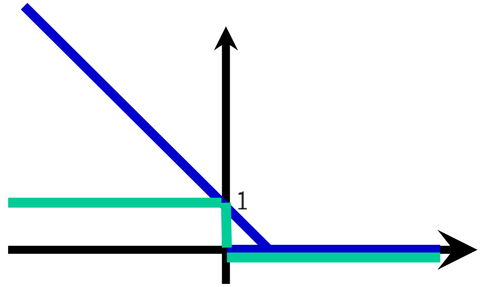
<u>Find</u> $\text{argmin}_{w,\xi_1,\ldots,\xi_m} ||w||^2 + C \sum_i \xi_i$ s.t.:

- For all i, $y_i w \cdot x_i \geq 1 - \xi_i$
  $\xi_i \geq 0$



ξ_i are "slack variables"

C controls the relative weighting between the twin goals of making the $||w||^2$ small (margin is large) and ensuring that most examples have functional margin ≥ 1.



$l(w, x, y) = \max(0, 1 - y\, w \cdot x)$

What you should know

- Perceptron simple online algo for learning linear separators with good guarantees when data has large geometric margin.

- The importance of margins in machine learning.

- The Support Vector Machines (SVM) algorithm.