Introduction to Machine Learning (Lecture Notes)

Multi-layer Perceptron

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1 The Multi-layer Perceptron

1.1 Matlab demos

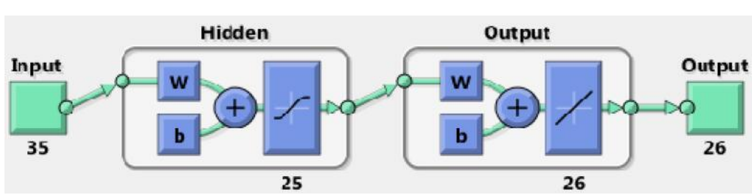
Matlab tutorials for neural network design:

```
nnd9sd % Steepest descent
nn9sdq % Steepest descent for quadratic
```

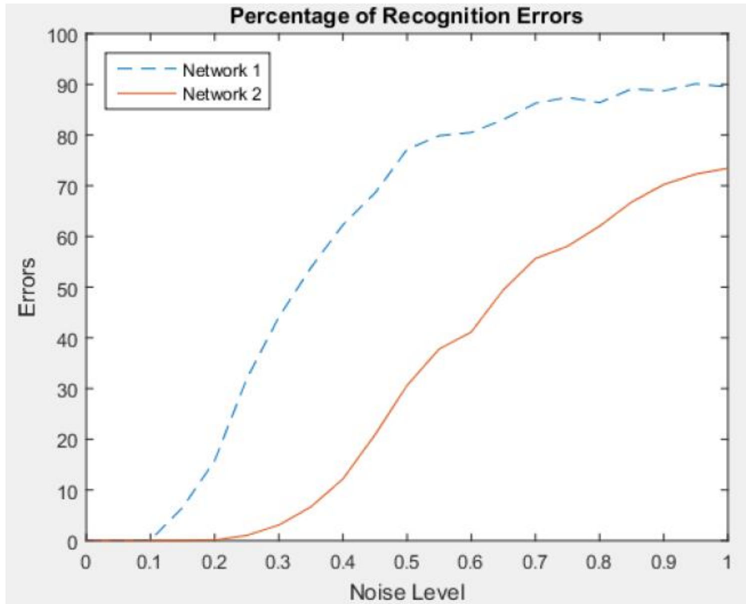
Character recognition with MLP:

```
appcr1
```

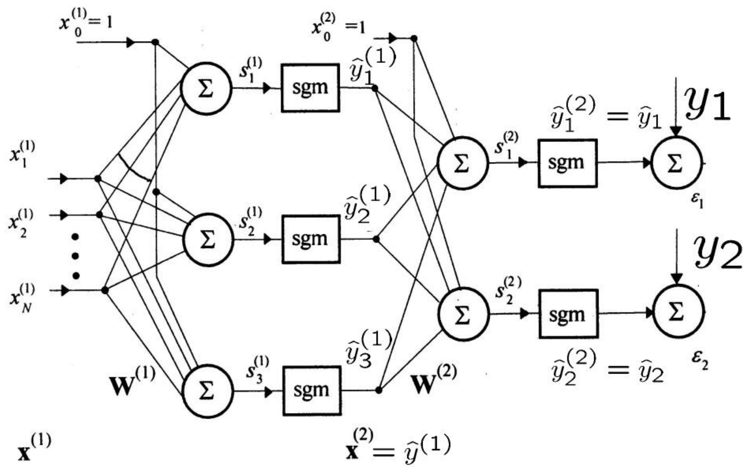
Struture of MLP:



Noise-free input: 26 different letters of size 7×5. Prediction errors:



1.2 An example and notations



Here we will always assume that the activation function is differentiable. This will allow us to optimize the cost function with gradient descent. However, non-differentiable activation functions are getting popular as well.

2 The back-propagation algorithm

2.1 The gradient of the error

The current error:

$$\epsilon^2 = \epsilon_1^2 + \epsilon_2^2 = (\hat{y}_1 - y_1)^2 + (\hat{y}_2 - y_2)^2.$$

More generally:

$$\epsilon^2 = \sum_{p=1}^{N_L} \epsilon_p^2 = \sum_{p=1}^{N_L} (\hat{y}_p - y_p)^2.$$

We want to calculate

$$\frac{\partial \epsilon(k)^2}{\partial W_{ij}^l(k)} = ?.$$

2.2 Notation

- $W_{ij}^l(k)$: At time step k, the strength of connection from neuron j on layer $l-1$ to neuron i on layer l. $(i = 1, 2, ..., N_l, j = 1, 2, ..., N_{l-1})$
- $s_i^l(k)$: The summed input of neuron i on layer l before applying the activation function f at time step $k (i = 1, ..., N_l)$.
- $x^l(k) \in \mathbb{R}^{N_{l-1}}$: The input of layer l at time step k.
- $\hat{y}^l(k) \in \mathbb{R}^{N_l}$: The output of layer l at time step k.
- $N_1, N_2, ..., N_l, ..., N_L$: Number of neurons in layers $1, 2, ..., l, ..., L$.

2.3 Some observations

$$x^l = \hat{y}^{l-1} \in \mathbb{R}^{\mathbb{N}_{l-1}}$$

$$s_i^l = W_{i.}^l \hat{y}^{l-1} = \sum_{j=1}^{N_{l-1}} W_{ij}^l x_j^l = \sum_{j=1}^{N_{l-1}} W_{ij}^l f(s_j^{l-1})$$

$$s_j^{l+1} = \sum_{i=1}^{N_l} W_{ji}^{l+1} f(s_i^l)$$

2.4 The back propagated error

Recall that $\frac{\partial}{\partial x} f(g(x), h(x)) = \frac{\partial}{\partial g} f(g(x), h(x)) \frac{\partial g(x)}{\partial x} + \frac{\partial}{\partial h} f(g(x), h(x)) \frac{\partial h(x)}{\partial x}$.

Introduce the notation:

$$\delta_i^l(k) = \frac{-\partial \epsilon^2(k)}{\partial s_i^l(k)} = -\sum_{p=1}^{N_L} \frac{\partial \epsilon_p^2(k)}{\partial s_i^l(k)}$$

where $i = 1, 2, ..., N_l$.

As a special case, we have that

$$\delta_i^L(k) = -\sum_{p=1}^{N_L} \frac{\partial (y_p(k) - f(s_p^L(k)))^2}{\partial s_i^L(k)} = 2\epsilon_i(k) f'(s_i^L(k))$$

Lemma 1 *$\delta_i^l(k)$ can be calculated from $\{\delta_1^{l+1}(k), ..., \delta_{N_{l+1}}^{l+1}(k)\}$ using backward recursion.*

$$\begin{aligned}\delta_i^l(k) &= -\sum_{p=1}^{N_L} \frac{\partial \epsilon_p^2}{\partial s_i^l} = \sum_{p=1}^{N_L} \sum_{j=1}^{N_{l+1}} -\frac{\partial \epsilon_p^2}{\partial s_j^{l+1}} \frac{\partial s_j^{l+1}}{\partial s_i^l} \\ &= \sum_{j=1}^{N_{l+1}} \sum_{p=1}^{N_L} -\frac{\partial \epsilon_p^2}{\partial s_j^{l+1}} W_{ji}^{l+1} f'(s_i^l)\end{aligned}$$

Therefore,

$$\delta_i^l(k) = (\sum_{j=1}^{N_{l+1}} \delta_j^{l+1} W_{ji}^{l+1}(k)) f'(s_i^l(k))$$

where $\delta_i^l(k)$ is the back propagated error.

Now using that

$$s_i^l(k) = \sum_{j=1}^{N_{l-1}} W_{ij}^l(k) x_j^l(k)$$

$$\frac{\partial \epsilon(k)^2}{\partial W_{ij}^l(k)} = \frac{\partial \epsilon(k)^2}{\partial s_i^l(k)} \frac{\partial s_i^l(k)}{\partial W_{ij}^l(k)} = -\delta_i^l(k) x_j^l(k)$$

The Back-propagation algorithm:

$$W_{ij}^l(k+1) = W_{ij}^l(k) + \mu \delta_i^l(k) x_j^l(k)$$

In vector form:

$$W_{i\cdot}^l(k+1) = W_{i\cdot}^l(k) + \mu \delta_i^l(k) x_{\cdot}^l(k).$$