Notes on basic probability and statistics

Yang Xu

1 A perspective on common distributions

1.1 Entropy (Shannon)

Entropy: average bits required to encode the uncertainty associated with \( X \),
e.g. the toss of a fair six-sided die contains between 2 and 3 bits of uncertainty,
or information.

\[
H(X) = \sum_i P(X_i) \log \frac{1}{P(X_i)}
\]  

(1)

1.2 Maximum entropy (Jaynes)

Maximum entropy: assignment of a distribution that assumes no more than is
known a priori. Which \( P(X) \) maximizes uncertainty in Equation 1, assuming
\( \sum_i P(X_i) = 1 \), which follows from the definition of a distribution, and nothing
else?

\[
L = H(X) + \lambda(\sum_i P(X_i) - 1)
\]  

(2)

1.2.1 Uniform distribution

Maximizing the Langrangian in Equation 2 with respect to \( P(X) \) and \( \lambda \) yields
the discrete uniform distribution, i.e. \( P(X) = \frac{1}{6} \) in the case of a six-sided die.
The continuous uniform distribution can be derived likewise.

\[
\frac{\partial L}{\partial P(X_j)} = -\log P(X_j) - 1 + \lambda = 0
\]  

(3)

\[
\frac{\partial L}{\partial \lambda} = \sum_i P(X_i) - 1 = 0
\]  

(4)

1.2.2 Gaussian distribution

The Gaussian distribution can also be derived from the maximum entropy prin-
ciple, under the added assumptions that the first (mean \( \mu \)) and second (variance
$\sigma^2$) order statistics are known, and with the entropy defined over a continuous distribution $H'(X) = \int P(X) \log \frac{1}{P(X_i)}$.

$$L' = H'(X) + \lambda \int P(X) - 1 + \alpha \int XP(X) - \mu + \beta \int (X - \mu)^2 P(X) - \sigma^2$$ (5)

### 1.2.3 Multivariate Gaussian distribution

The multivariate Gaussian distribution is a generalization of the univariate version to arbitrary dimensions. The devil is in the covariance matrix (derivation of the 2-D case omitted).

## 2 Two basic laws of asymptotics

### 2.1 The law of large numbers

LLN: convergence of sample average toward the expectation.

### 2.2 The central limit theorem

CLT: convergence of form of aggregated samples toward the normal distribution.