10-751 Final Exam/Assignment/Mini-Project

This is the final exam for the class. Rather than a traditional exam, it is more of a mini-project to explore some interesting aspects of Markov chains, exercising some basic concepts in probability and statistics. There are five problems.

You have until 5:00 p.m. Monday, December 9 to hand in your solutions to the problems. Please hand in your written solutions to Wean 4111 by this time.

Good luck, and have fun!

Problem 1. Consider the following gambling game, which we will refer to as *Game* A_ϵ. You start with a total holdings of $\$X$ (an integer). On each play of the game, you win \$1 with probability $\frac{1}{2} - \epsilon$, and you lose \$1 with probability $\frac{1}{2} + \epsilon$, where $\epsilon > 0$ is a small positive number. Assume that the game continues if your holdings are negative. (This just means that you owe me money:-)

1. Describe the underlying probability space for this game.
2. What is your expected holdings after the first play of the game?
3. What is the variance of your holdings after the first play of the game?
4. What is your expected holdings after n plays of the game?
5. Give the full probability mass function for your holdings after n plays of the game.

Problem 2. Continuing with Game A_ϵ, write a Matlab program that simulates the game. Let $X(n)$ be your holdings after n plays of the game.

1. Run five separate simulations of 100 steps of game $A_{0.01}$, and plot the results in five separate plots of $X(n)$, $n = 0, 1, \ldots, 100$.
2. Run 1,000 games, and *average* the results to get $\overline{X}(n)$. Plot the result.
3. Include a *printout* of your Matlab program. (Do not pass in the source code.)

You may wish to try different values of ϵ to explore the effect of this parameter on the game.

Problem 3. Now consider a slightly more interesting game, which we will refer to as *Game* B_ϵ. Again, you start with total holdings \$$X$ (an integer). Under this game, if your holdings is a multiple of 3, then you win \$1 with probability $\frac{1}{10} - \epsilon$ and lose \$1 with probability $\frac{9}{10} + \epsilon$. However, if your current holdings *is not* a multiple of 3, then you win \$1 with probability $\frac{3}{4} - \epsilon$ and you lose \$1 with probability $\frac{1}{4} + \epsilon$.

1. Consider the following argument that for small ϵ, B_ϵ is a winning game (that is, you will on average come out ahead):

   Since there is a $\frac{1}{3}$ chance that $X(n) = 0 \ (mod\ 3)$, the probability of winning is $\frac{1}{3}(\frac{1}{10} - \epsilon) + \frac{2}{3}(\frac{3}{4} - \epsilon) = \frac{8}{15} - \epsilon$, which is larger than $\frac{1}{2}$ for small ϵ.

   What is wrong with this argument?

2. Express this game in terms of a Markov chain. Clearly state the state space and transition matrix for the Markov chain.

3. What is the stationary distribution of the Markov chain you obtained above for $\epsilon = 0.01$?

4. Use the Markov chain formulation to show that the game B_ϵ is in fact *not* a winning game for small $\epsilon > 0$.

Problem 4. Continuing with Game B_ϵ, write a Matlab program that simulates the game. Let $X(n)$ be your holdings after n plays of the game.

1. Run five separate simulations of 100 steps of game $B_{0.01}$, and plot the results in five separate plots of $X(n)$, $n = 0, 1, \ldots, 100$.

2. Run 1,000 games, and *average* the results to get $\overline{X}(n)$. Plot the result.

3. Again, include a *printout* of your Matlab program. (Do not pass in the source code.)

You may again wish to try different values of ϵ to explore the effect of this parameter on the game.

Problem 5. Now things get interesting. Consider next a game that alternates between plays of A_ϵ and B_ϵ. Precisely, let $C_{\epsilon,k}$ be the game where k plays of A_ϵ are followed by k plays of B_ϵ, followed by k more plays of A_ϵ, k more plays of B_ϵ, and so on. Let $X(n)$ be your holdings after n plays of this new game.

1. Run five separate simulations of 100 steps of $C_{0.01,2}$, and plot the results.

2. Now run 5,000 (or more) games of $C_{0.01,2}$, and *average* the results to get $\overline{X}(n)$. Plot the result.

3. What is going on? How have two losing games been turned into a winning game? Experiment with different settings of k and ϵ. Can you explain why the behavior of $C_{\epsilon,k}$ is so different from that of A_ϵ and B_ϵ? Can you predict what will happen when a random choice of playing A_ϵ or B_ϵ is made on each turn?