10-751 First Exam

There are five problems on the exam, each worth 20 points. Write your solutions on the attached pages, clearly marking which problem your solution applies to. You have one hour and twenty minutes to work the problems. Books and notes are allowed, but no computers.

Problem 1.

1. Suppose that X is a random variable with cumulative distribution function (cdf) F. What is the cdf of the random variable $Y = aX + b$, where a and b are constants?
2. Suppose that F and G are cdfs. Which of the following are also cdfs of some random variable?
   (a) $\lambda F + (1 - \lambda) G$, where $0 \leq \lambda \leq 1$.
   (b) FG.
   (c) F/G
   (d) $F + G$
   (e) F^k for a positive integer k.
   (f) $1 - (1 - F)^k$ for a positive integer k.

Problem 2. Two fair dice are rolled. Let X denote the number of dots appearing on the first of the dice, and let S denote the sum of the dots on the two dice. For which k is the event $\{S = k\}$ independent of X?

Problem 3. Let X, Y, and Z be random variables. For each of the following equalities, either prove that it is correct or derive a counterexample showing that it does not in general hold.

1. If X and Y are independent, then $E(Y \mid X) = E(Y)$.
2. $E(g(X)\, Y \mid X) = g(X)\, E(Y \mid X)$ for any function g.
3. $E(E(Y \mid X, Z) \mid X) = E(Y \mid X)$.

To simplify things, you may assume that X, Y, and Z are discrete.

Problem 4. Let X and Y be independent Poisson random variables with parameters λ and μ respectively.

1. What is the parametric form of the distribution of $X + Y$?
2. What is the parametric form of the distribution of X conditioned on the event $X + Y = n$?

Explain your answers.

Problem 5. Two urns contain a total of N balls. Each second, one of the N balls is chosen at random and moved to the other urn, forming a random walk with $N + 1$ states s_k, where k is the number of balls in the first urn. For $N = 4$, the resulting transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

1. Show that $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = \left(\frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}\right)$ is a stationary distribution for this Markov chain.
2. Determine the probability of, starting in state 0, reaching state 4 before returning to state 0 (for $N = 4$). You may give your answer as a matrix expression, it's not necessary to calculate a numerical value.