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Multi-Heuristic A*

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Abstract

The performance of heuristic search based planners depends heavily on the quality of the heuristic function used to focus the search. These algorithms work fast and generate high-quality solutions, even for high-dimensional problems, as long as they are given a well-designed heuristic function. On the other hand, their performance can degrade considerably if there are large heuristic depression regions, i.e., regions in the search space where heuristic values do not correlate well with the actual cost-to-goal values. Consequently, the research in developing an efficient planner for a specific domain becomes the design of a good heuristic function. However, for many domains, it is hard to design a *single* heuristic function that captures all the complexities of the problem. Furthermore, it is hard to ensure that heuristics are admissible (provide lower bounds on the cost-to-goal) and consistent, which is necessary for A* like searches to provide guarantees on completeness and bounds on sub-optimality. In this paper, we develop a novel heuristic search, called Multi-Heuristic A* (MHA*), that takes in *multiple, arbitrarily inadmissible* heuristic functions in addition to a single consistent heuristic, and uses all of them simultaneously to search in a way that preserves guarantees on *completeness* and *bounds on sub-optimality*. This enables the search to combine very effectively the guiding powers of different heuristic functions and simplifies dramatically the process of designing heuristic functions by a user because these functions no longer need to be admissible or consistent. We support these claims with experimental analysis on several domains ranging from inherently continuous domains such as full-body manipulation and navigation to inherently discrete domains such as the sliding tile puzzle.

1. Introduction

Heuristic search algorithms (such as A* (Hart et al., 1968)) have been a popular choice for low-dimensional path planning in robotics since the 1980s because of their intuitive appeal and guarantees of completeness and optimality. These methods take advantage of admissible heuristics (optimistic estimates of cost-to-go) to speed up search. However, for high-dimensional planning problems, heuristic search methods suffer from the curse of dimensionality. In the last decade, researchers have addressed this problem by trading off solution quality for faster computation times. Specifically, bounded sub-optimal versions of A* such as Weighted A* (WA*) (Pohl, 1970) and its anytime variants (Likhachev et al., 2004; Zhou and Hansen, 2002) have been used quite effectively for high-dimensional planning problems in robotics ranging from motion

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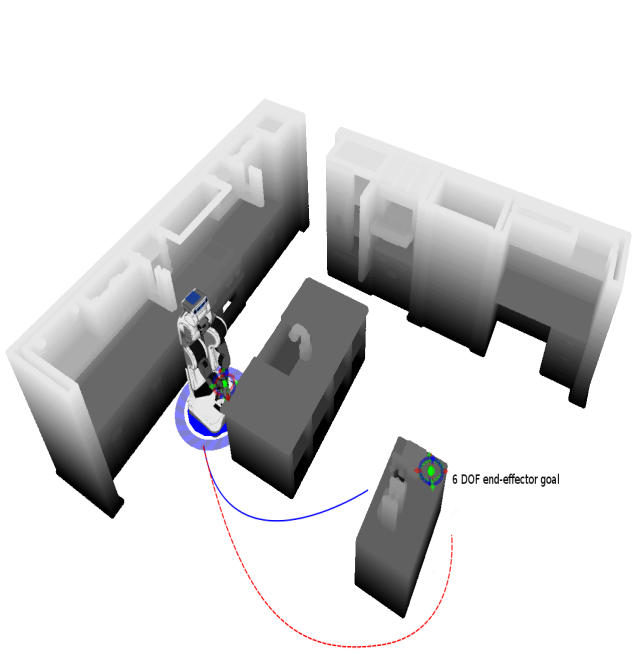
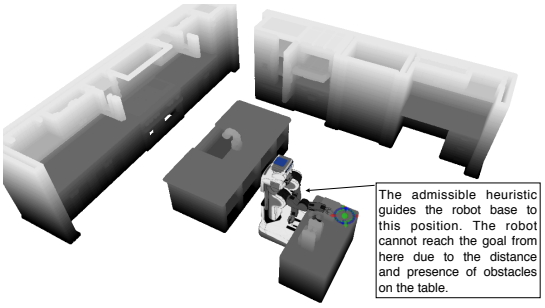
planning for ground vehicles (Likhachev and Ferguson, 2009) and flight planning for aerial vehicles (MacAllister et al., 2013) to planning for manipulation (Cohen et al., 2013) and footstep planning for humanoids (Hornung et al., 2013) and quadrupeds (Zucker et al., 2011). All of these planners achieve faster speeds than A* search by inflating the heuristic values with an inflation factor ($w > 1$) to give the search a depth-first flavor. They also provide bounds on the solution sub-optimality, namely, the factor (w) by which the heuristic is inflated (Pearl, 1984).

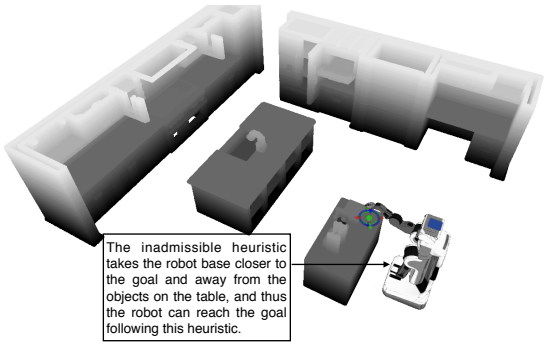
As such though, these algorithms rely greatly on the guiding power of the heuristic function. In fact, WA*'s performance is very sensitive to the accuracy of the heuristic function, and can degrade severely in the presence of heuristic depression regions, i.e., regions in the search space where the correlation between the heuristic values and the actual cost-to-go is weak (Hernández and Baier, 2012; Wilt and Ruml, 2012). WA* can easily get trapped in these regions as its depth-first greediness is guided by the heuristic values, and may require expanding most of the states belonging to a depression zone before exiting. Presence of large depression zones is common in path planning domains (especially in cluttered environments) as the heuristics are typically computed by solving a relaxed version of the problem (such as by ignoring kinodynamic constraints). Consequently, the depth-first path suggested by the heuristic function may not be feasible in reality, leading the search into a "local minimum". Therefore, designing heuristic functions that are admissible, consistent, and have shallow depression regions remains challenging for complex planning problems.



(a) Heuristics



(b) Greedy search with the admissible heuristic



(c) Greedy search with the inadmissible heuristic

Fig. 1. A full-body (12D: x,y,orientation for the base + spine + 6 DOF object pose + 2 free angles for the arms) manipulation planning (with PR2) example depicting the utility of multiple heuristics. Figure 1a shows two heuristic paths that correspond to greedily following two different heuristics, solid curve for an admissible heuristic and dotted curve for an inadmissible heuristic. Greedily following the admissible heuristic guides the search to a depression region where the search gets stuck (Figure 1b). In contrast, greedily following the inadmissible heuristic guides the search along a feasible path and therefore allows the planner to efficiently compute a valid plan (Figure 1c).

In contrast, for many domains, one can easily compute different inadmissible heuristics each of which can provide complementary guiding power. For example, in Figure 1 we include a full-body manipulation scenario where the PR2

robot needs to grasp an object on the table, marked by "end-effector goal". The admissible heuristic function (path shown by the solid curve, Figure 1a) guides the search to a local minimum as the robot cannot reach the object from the left side of the table (Figure 1b). However, we can obtain multiple inadmissible heuristics by computing the navigation (x, y) distance to different points around the object to be grasped. In the example (Figure 1a), we show one such additional inadmissible heuristic function that guides the search through a different route (shown by the dotted curve to the right side of the table). Using this heuristic, the search does find a pose that allows the robot to grasp the object, i.e., it computes a valid plan (Figure 1c).

When used in isolation, these additional heuristics provide little value, as they can neither guarantee completeness (because they can be arbitrarily inadmissible), nor can they guarantee efficient convergence (because each may have its own depression region). However, as shown in this paper, a search can consider multiple such hypotheses to explore different parts of the search space while using a consistent heuristic to ensure completeness. This may result in faster convergence if any of these heuristics (or a combination) can effectively guide the search around the depression regions. We present an algorithmic framework, called Multi-Heuristic A* (MHA*), that builds on this observation. MHA* works by running multiple searches with different inadmissible heuristics in a manner that preserves completeness and guarantees on the sub-optimality bounds.

We propose two variants of MHA*: Independent Multi-Heuristic A* (IMHA*) which uses independent g and h values for each search, and Shared Multi-Heuristic A* (SMHA*) which uses different h values but a single g value for all the searches. We show that with this shared approach, SMHA* can guarantee the sub-optimality bounds with at most two expansions per state. In addition, SMHA* is potentially more powerful than IMHA* in avoiding depression regions as it can use a combination of partial paths found by different searches to reach the goal.

We discuss the theoretical properties of MHA* algorithms, proving their completeness, sub-optimality bounds and state re-expansions bounds. We present experimental results for two robotics domains, namely, 12D mobile manipulation for PR2 (full-body) and 3D (x, y, orientation) navigation. We also show that MHA* can be a natural choice when solving multiple goal planning problems and include experimental results to support this observation. These experiments demonstrate the efficacy of MHA*, especially for cases where the commonly used heuristics do not lead the search well. We also include experimental results for large sliding tile puzzle problems, highlighting the benefits of the proposed framework outside of robotics domains[1].

2. Related Work

The utility of having multiple heuristics has been noted in many search applications including motion planning (Likhachev and Ferguson, 2009), searching with pattern database heuristics (Felner et al., 2004; Korf and Felner, 2002), AND/OR graphs (Chakrabarti et al., 1992) etc. For example, in (Likhachev and Ferguson, 2009) the maximum of two admissible heuristics (one mechanism-relative and another environment-relative) was used, as it could guide the planner better when compared to the individual heuristics. In (Felner et al., 2004), it was shown that a more informative heuristic function can be created by adding multiple disjoint pattern database heuristics, and such a heuristic can substantially enhance search performance. The key difference between these approaches and ours is that while these utilize multiple heuristics, the information is combined to create a single best (often consistent) heuristic value, which is then used to guide the search. In contrast, MHA* uses multiple heuristics independently to explore different regions of the search space. Also, as we derive the bounds using a consistent heuristic, the additional heuristics can be arbitrarily inadmissible, which makes them easy to design.

[1] An initial version of this work appeared in (Aine et al., 2014). In this paper we provide a complete report on MHA* algorithms with detailed discussions of their theoretical properties as well as experimental results.

The idea of deriving the bounds based on a consistent heuristic and using inadmissible estimates/constraints to guide the search has also been used in several other search algorithms (Aine et al., 2007; Chakrabarti et al., 1989; Pearl and Kim, 1982; Thayer and Ruml, 2011; Thayer et al., 2012). For example, the A^*_ϵ algorithm (Pearl and Kim, 1982) uses a distance-to-go estimate to determine the order of expansion among the states whose f values (computed using a consistent heuristic) lie within the chosen bound of the minimum f value in the open list. In the bounded quality version of Anytime Window A* (Aine et al., 2007), the search is confined within a window of fixed size, as long as the f values do not exceed the bound on the minimum f value among the suspended states (states that are outside the current window). A more recent algorithm, Explicit Estimation Search (EES (Thayer and Ruml, 2011)), uses the same principle as A^*_ϵ to satisfy the bounds, but improves the performance by using an additional inadmissible distance function to guide the search. These algorithms show how we can use arbitrary estimates to guide the search and yet compute bounded sub-optimal solutions. While MHA* follows a similar philosophy to derive the sub-optimality bounds, the concept of simultaneous searching with multiple heuristics sets it apart. Also, all the above mentioned algorithms require unrestricted re-expansion of states to satisfy the bounds, in contrast, MHA* provides a bound on the maximum number of re-expansions possible.

While, the idea of exploring multiple heuristics independently (in contrast to combining them with sum, max etc) has been there for a long time (Chakrabarti et al., 1989), in (Helmert, 2006), an algorithm was proposed that *s*imultaneously uses multiple heuristics to search. The proposed algorithm, Multiple Heuristic Greedy Best-first Search (MHGBFS) explores the search space in a greedy best-first manner (Doran and Michie, 1966) using multiple queues, each of which uses a different heuristic as the priority. Once a state is expanded, all its successors are evaluated by all heuristics, and put into every queue with the corresponding value. In (Röger and Helmert, 2010), an empirical examination was performed on how to exploit multiple heuristics for satisficing planning, among the different benchmarked approaches, the above discussed method (called the alternation method in (Röger and Helmert, 2010)) was shown to be the best. In (Isto, 1996), a robotics path planning algorithm was proposed that utilizes multiple heuristics and attempts to share the resource among individual searches. The basic philosophy of these algorithms is similar to the MHA* approach, however, they do not provide any guarantees on completeness/sub-optimality. In contrast, because we introduce a consistent heuristic search to anchor the explorations, MHA* guarantees a) completeness, b) bounded sub-optimality and c) bounded expansions even with arbitrarily inadmissible heuristics.

(Valenzano et al., 2010) proposed simultaneous searching with different parameters (such as operator orders, heuristic weights), as an alternative to fine tuning the parameters. This method was shown to be effective for several problems especially when resources are available for parallel exploration. However, this framework also relies on a single heuristic to guide the search (albeit with different parameters) and therefore can suffer similarly when the heuristic has large depression regions.

A completely different approach to path planning is adopted by the sampling-based planners (Karaman and Frazzoli, 2010; Kavraki et al., 1996; Lavalle et al., 2000). The fundamental difference between the sampling and heuristic search based planners is that the sampling based algorithms primarily target continuous spaces whereas the search algorithms are for planning in discrete spaces, independent of whether these came as the result of discretizing the state-space or from an inherently discrete system. Moreover, most sampling-based planners (with the exception of the RRT* algorithm (Karaman and Frazzoli, 2010)) focus on finding any feasible trajectory, rather than minimizing the cost of the solution, and therefore can produce arbitrarily sub-optimal and unpredictable solutions. In contrast, heuristic search based planning methods often provide better cost minimization and more consistent behavior (compared to the sampling-based planners), but at the expense of higher planning times and need for a well-designed heuristic function. Our work targets the last issue as we try to alleviate the dependency on having a single well-designed heuristic function by supporting multiple heuristic functions that can be arbitrarily inadmissible.

3. Multi-Heuristic A*

In this section, we describe two multi-heuristic search algorithms, IMHA* and SMHA*, and discuss their properties.

Notations and Assumptions : In the following, S denotes the finite set of states of the domain. $c(s, s')$ denotes the cost of the edge between s and s', if there is no such edge, then $c(s, s') = \infty$. We assume $c(s, s') \geq 0$, $\forall s, s'$ pairs. Succ$(s) := \{s' \in S | c(s, s') \neq \infty\}$, denotes the set of all successors of s. $c^*(s, s')$ denotes the cost of the optimal path from state s to s'. $g^*(s)$ denotes the optimal path cost from s_{start} to s. $g(s)$ denotes the currently best known path cost from s_{start} to s, and $bp(s)$ denotes a back-pointer which points to the best predecessor of s (if computed).

We assume that we have n heuristics denoted by h_i for $i = 1, 2, \ldots, n$. These heuristics do not need to be consistent, in fact, they can be arbitrarily inadmissible. We also require access to a consistent (and thus admissible) heuristic (denoted by h_0), i.e., h_0 should satisfy, $h_0(s_{goal}) = 0$ and $h_0(s) \leq h_0(s') + c(s, s')$, $\forall s, s'$ pair, where $s' \in$ Succ(s) and $s \neq s_{goal}$ (Pearl, 1984). MHA* uses separate priority queues for each search ($n + 1$ queues), denoted by OPEN$_i$, for $i = 0..n$. Throughout the rest of the paper, we will use the term *anchor* search to refer to the search that uses h_0. Other searches will be referred to as *inadmissible* searches. We assume that each priority queue (OPEN$_i$) supports a function OPEN$_i$.Minkey(), which returns the minimum key value among the states present in the priority queue if the queue is not empty, else it returns ∞.

3.1. Independent Multi-Heuristic A (IMHA*)*

The algorithm IMHA* is presented in Algorithm 1. In IMHA*, different heuristics are explored *independently* by simultaneously running $n + 1$ searches, where each search uses its own priority queue. Therefore, in addition to the different h values, each state uses a different g (and *bp*) value for each search. We use g_0 to denote the g for the anchor search, and g_i ($i = 1, 2, \ldots, n$) for the other searches.

The sub-optimality bound is controlled using two variables, namely, $w_1 (\geq 1.0)$ which is used to inflate the heuristic values for each of the searches, and $w_2 (\geq 1.0)$ which is used as a factor to prioritize the inadmissible searches over the anchor search. IMHA* runs the inadmissible searches in a round robin manner in a way which guarantees that the solution cost will be within the sub-optimality bound $w_1 * w_2$ of the optimal solution cost.

IMHA* starts with initializing search variables (lines 13-18) for all the searches. It then performs best-first expansions in a round robin fashion from queues OPEN$_i$ $i = 1..n$, as long as OPEN$_i$.Minkey() $\leq w_2 *$ OPEN$_0$.Minkey() (line 21). If the check is violated for a given search, it is suspended and a state from OPEN$_0$ is expanded in its place. This in turn can increase OPEN$_0$.Minkey() (lower bound) and thus re-activate the suspended search[2].

Expansion of a state is done in a similar way as done in A*. Each state is expanded at most once for each search (line 10) following the fact that WA* does not need to re-expand states to guarantee the sub-optimality bound (Likhachev et al., 2004). IMHA* terminates successfully, if any of the searches have OPEN$_i$.Minkey() value greater than or equal to the g value of s_{goal} (in that search) and $g(s_{goal}) < \infty$, otherwise it terminates with no solution when OPEN$_0$.Minkey() $\geq \infty$.

Next, we discuss the theoretical properties of IMHA*. First, we note that the anchor search in IMHA* is a single shot WA* (without re-expansions) with a consistent heuristic function h_0 (as used in (Likhachev et al., 2004; Richter et al., 2010)). Thus, all the results of such a WA* are equally applicable in the case of the anchor search in IMHA*. We start by presenting two key theorems for the anchor search which shall later be used to derive the properties of IMHA*. At this point, we introduce two assumptions that will be used for the proofs through out the paper. First, we assume that if

[2] It should be noted that an expansion from OPEN$_0$ can also decrease the lower bound and thus suspend more inadmissible searches. One admissible way to get rid of this is to allow the lower bound to only increase, i.e., we store the initial value of OPEN$_0$.Minkey() in a variable and later only update it if OPEN$_0$.Minkey() increases.

Algorithm 1 Independent Multi-Heuristic A* (IMHA*)

```
1: procedure Key(s, i)
2:     return g_i(s) + w_1 * h_i(s)
3: procedure ExpandState(s, i)
4:     Remove s from OPEN_i
5:     for all s' ∈ Succ(s) do
6:         if s' was never generated in the i^th search then
7:             g_i(s') = ∞; bp_i(s') = null
8:         if g_i(s') > g_i(s) + c(s, s') then
9:             g_i(s') = g_i(s) + c(s, s'); bp_i(s') = s
10:            if s' ∉ CLOSED_i then
11:                Insert/Update s' in OPEN_i with Key(s', i)
12: procedure Main()
13:    for i = 0, 1, ..., n do
14:        OPEN_i ← ∅
15:        CLOSED_i ← ∅
16:        g_i(s_start) = 0; g_i(s_goal) = ∞
17:        bp_i(s_start) = bp_i(s_goal) = null
18:        Insert s_start in OPEN_i with Key(s_start, i)
19:    while OPEN_0.Minkey() < ∞ do
20:        for i = 1, 2, ..., n do
21:            if OPEN_i.Minkey() ≤ w_2 * OPEN_0.Minkey() then
22:                if g_i(s_goal) ≤ OPEN_i.Minkey() then
23:                    if g_i(s_goal) < ∞ then
24:                        Terminate and return path pointed by bp_i(s_goal)
25:                else
26:                    s ← OPEN_i.Top()
27:                    ExpandState(s, i)
28:                    Insert s in CLOSED_i
29:            else
30:                if g_0(s_goal) ≤ OPEN_0.Minkey() then
31:                    if g_0(s_goal) < ∞ then
32:                        terminate and return path pointed by bp_0(s_goal)
33:                else
34:                    s ← OPEN_0.Top()
35:                    ExpandState(s, 0)
36:                    Insert s in CLOSED_0
```

OPEN$_i = \emptyset$ ($\forall i = 0, 1, \ldots, n$), then its minimim key value is ∞. We also assume that any state s with undefined g_i and bp_i values (s not generated in i^{th} search) has $g_i(s) = \infty$ and $bp_i = \textbf{null}$[3].

Theorem 1. *At line 20, for any state s with* Key$(s, 0) \leq$ Key$(u, 0)$ $\forall u \in$ *OPEN*$_0$*), it holds that* $g_0(s) \leq w_1 * g^*(s)$.

Proof. We borrow this Theorem from the results described in (Likhachev et al., 2004), which states that the g value (g_0 here) for any state to be expanded in WA* (anchor search here) is at most w_1-sub-optimal. For detailed proof, please see (Likhachev et al., 2003). □

Theorem 2. *At line 20, it holds that* *OPEN*$_0$.Minkey$() \leq w_1 * g^*(s_{goal})$. *Where* *OPEN*$_0$.Minkey$() =$ ***Min***(Key$(u, 0)$ $\forall u \in$ *OPEN*$_0$*) if* *OPEN*$_0 \neq \emptyset$*,* ∞ *otherwise.*

Proof. We assume that the $g^*(s_{goal}) < \infty$, as otherwise the theorem holds trivially.

We prove this by contradiction. Let us say that $M =$ OPEN$_0$.Minkey(). First, we assume that $M > w_1 * g^*(s_{goal})$, then we will show that in such a case, there is at least one state $s' \in$ OPEN$_0$ having Key$(s', 0) < M$ arriving at a contradiction to the statement that $M =$ OPEN$_0$.Minkey().

Let us consider a least cost path from s_{start} to s_{goal} given as $P = \Pi(s_0 = s_{start}, ..., s_k = s_{goal})$. From this path, we pick the first state s_i that has not yet been expanded by the anchor search, but is part of OPEN$_0$ ($s_i \in$ OPEN$_0$). Note that, we will always find such a state $s_i \in$ OPEN$_0$ because a) $s_0 = s_{start}$ is put in OPEN$_0$ at the initialization (line 18), b) whenever any state $s_j \in P$ is expanded in the anchor search (i.e., removed from OPEN$_0$), $s_{j+1} \in P$ is always inserted in OPEN$_0$, and c) $s_k = s_{goal}$ is never expanded in the anchor search, as whenever s_{goal} has the least key value in OPEN$_0$ (i.e., $g_0(s_{goal}) \leq$ OPEN$_0$.Minkey()) the search terminates (line 32). Note that, the presence of such a state s_i also ensures that OPEN$_0$ is never empty if there is a finite cost solution.

Now, let us examine $g_0(s_i)$. If $i = 0$, we have $g_0(s_i) = g_0(s_{start}) = 0 \leq w_1 * g^*(s_i)$ (as, $g^*(s_{start}) = g_0(s_{start}) = 0$). If $i \neq 0$, by the choice of s_i we know that s_{i-1} has already been expanded in the anchor search. When s_{i-1} was chosen for expansion, we had $g_0(s_{i-1}) \leq w_1 * g^*(s_{i-1})$ from Theorem 1 and g_i value of a state is never increased during the execution of IMHA*. Now, as s_i is a successor of s_{i-1}, we have

$$\begin{aligned} g_0(s_i) &\leq g_0(s_{i-1}) + c(s_{i-1}, s_i) \text{ (line 9, Algorithm 1)} \\ &\leq w_1 * g^*(s_{i-1}) + c(s_{i-1}, s_i) \\ &\leq w_1 * (g^*(s_{i-1}) + c(s_{i-1}, s_i)) \text{(As } s_{i-1}, s_i \in \text{optimal path)} \\ &= w_1 * g^*(s_i) \end{aligned} \tag{1}$$

Thus, we have $g_0(s_i) \leq w_1 * g^*(s_i)$. Using this we obtain,

$$\begin{aligned} \text{Key}(s_i, 0) &= g_0(s_i) + w_1 * h_0(s_i) \\ &\leq w_1 * g^*(s_i) + w_1 * h_0(s_i) \\ &\leq w_1 * g^*(s_i) + w_1 * c^*(s_i, s_{goal}) \; h_0 \text{ is consistent, thus admissible} \\ &\leq w_1 * g^*(s_{goal}) \end{aligned} \tag{2}$$

Now, as $s_i \in$ OPEN$_0$ and Key$(s_i, 0) \leq w_1 * g^*(s_{goal}) < M$, i.e., we have a contradiction to our assumption that $M =$ OPEN$_0$.Minkey$() > w_1 * g^*(s_{goal})$ (from the definition of OPEN$_0$.Minkey()). □

Next, we present three theorems summarizing the main properties of IMHA*.

Theorem 3. *When IMHA* exits (in the i^{th} search), $g_i(s_{goal}) \leq w_1 * w_2 * g^*(s_{goal})$, i.e., the solution cost obtained is bounded by $w_1 * w_2$ sub-optimality factor.*

[3] It may be noted that these assumptions are similar to those used in (Likhachev et al., 2004), for the proofs (details in (Likhachev et al., 2003)), which ensures the results used in (Likhachev et al., 2004) remains applicable for the anchor search.

Proof. IMHA* can terminate successfully in lines 32 (anchor search) or 24 (inadmissible search), or it can terminate without a solution in line 19.

If the anchor search terminates at line 32, i.e., Key$(s_{goal}, 0) \leq$ Key$(u, 0)$, $\forall u \in$ OPEN$_0$, from Theorem 1 we have,

$$\begin{aligned} g_0(s_{goal}) &\leq w_1 * g^*(s_{goal}) \\ &\leq w_1 * w_2 * g^*(s_{goal}) \\ &\text{As } w_2 \geq 1.0 \end{aligned} \tag{3}$$

If an inadmissible search (say i^{th}) terminates in line 24, then from lines 21 and 24, we have,

$$\begin{aligned} g_i(s_{goal}) &\leq w_2 * \text{OPEN}_0.\text{Minkey}() \\ &\leq w_2 * w_1 * g^*(s_{goal}) \text{ (From Theorem 2)} \end{aligned} \tag{4}$$

Therefore, in both the above mentioned cases, i.e., if either the anchor terminates or an inadmissible search terminates, we have the solution cost to be within $w_1 * w_2$ factor of the optimal solution cost.

On the other hand, if the search terminates unsuccessfully at line 19 (while condition is not satisfied), from Theorem 2 we know OPEN$_0$.Minkey$() \leq w_1 * g^*(s_{goal})$. Now, OPEN$_0$.Minkey$() \geq \infty \implies g^*(s_{goal}) \geq \infty$, i.e., there is no finite cost solution. □

Theorem 4. *No state is expanded more than $n + 1$ times during the execution of the IMHA*.*

Proof. In IMHA*, a state s can only be expanded when it is selected as the top state of OPEN$_i$ in either line 26 or 34. In both the cases the very next call is to the function ExpandState, which removes this selected state from OPEN$_i$ (line 4). Now, a state (other than s_{start}) can only be inserted in OPEN$_i$ in line 11. If a state s has already been expanded in the i^{th} search (i.e. $s \in$ CLOSED$_i$), the check at line 10 will ensure that s is not inserted again in OPEN$_i$, and therefore cannot be expanded in the i^{th} search any more. In other words, a state s can only be expanded *at most* once in every search, and as the total number of searches is $n + 1$ (anchor search and n inadmissible searches), no state is expanded more than $n + 1$ times. □

Theorem 5. *In IMHA*, a state s is never expanded in the i^{th} inadmissible search if* Key$(s, i) > w_1 * w_2 * g^*(s_{goal})$.

Proof. This theorem can be proved using Theorem 2, which states that OPEN$_0$.Minkey$() \leq w_1 * g^*(s_{goal})$. If a state s is selected for expansion in the i^{th} inadmissible search at line 26, it has Key$(s, i) \leq$ OPEN$_i$.Minkey$()$ (from the priority queue properties). Now, from the check at line 21, we obtain OPEN$_i$.Minkey$() \leq w_2 *$ OPEN$_0$.Minkey$()$, otherwise the i^{th} search will be suspended and the control will not reach line 26. Therefore, if a state s is expanded in the i^{th} search, we have

$$\begin{aligned} \text{Key}(s, i) &\leq \text{OPEN}_i.\text{Minkey}() \\ &\leq w_2 * \text{OPEN}_0.\text{Minkey}() \\ &\leq w_2 * w_1 * g^*(s_{goal}) \end{aligned} \tag{5}$$

□

Theorem 3 guarantees the sub-optimality bounds for IMHA* while Theorems 4 and 5 provide the bounds on state expansions by IMHA*. They also ensure that IMHA* is complete and guaranteed to terminate for a finite search space. The efficiency of IMHA* stems from the fact that it terminates as soon as any one of the n heuristics leads the search to a solution within the sub-optimality bound, i.e., the total state expansions $\approx n \times$ minimum of the state expansions among all the searches. Thus, if any of the inadmissible searches converges faster than the anchor search by a factor better than n, IMHA* can outperform WA*.

Algorithm 2 Shared Multi-Heuristic A* (SMHA*)

```
1: procedure Key(s, i)
2:     return g(s) + w_1 * h_i(s)
3: procedure ExpandState(s)
4:     Remove s from OPEN_i ∀ i = 0, 1, ..., n
5:     v(s) = g(s)                                          ▷ /* For proofs only. */
6:     for all s' ∈ Succ(s) do
7:         if s' was never generated then
8:             g(s') = ∞; bp(s') = null
9:             v(s') = ∞                                    ▷ /* For proofs only. */
10:        if g(s') > g(s) + c(s, s') then
11:            g(s') = g(s) + c(s, s'); bp(s') = s
12:            if s' ∉ CLOSED_anchor then
13:                Insert/Update s' in OPEN_0 with Key(s', 0)
14:                if s' ∉ CLOSED_inad then
15:                    for i = 1, 2, ..., n do
16:                        if Key(s', i) ≤ w_2 * Key(s', 0) then
17:                            Insert/Update s' in OPEN_i with Key(s', i)
18: procedure Main()
19:    g(s_start) = 0; g(s_goal) = ∞
20:    bp(s_start) = bp(s_goal) = null
21:    v(s_start) = v(s_goal) = ∞                           ▷ /* For proofs only. */
22:    for i = 0, 1, ..., n do
23:        OPEN_i ← ∅
24:        Insert s_start in OPEN_i with Key(s_start, i)
25:    CLOSED_anchor ← ∅
26:    CLOSED_inad ← ∅
27:    while OPEN_0.Minkey() < ∞ do
28:        for i = 1, 2, ..., n do
29:            if OPEN_i.Minkey() ≤ w_2 * OPEN_0.Minkey() then
30:                if g(s_goal) ≤ OPEN_i.Minkey() then
31:                    if g(s_goal) < ∞ then
32:                        Terminate and return path pointed by bp(s_goal)
33:                else
34:                    s ← OPEN_i.Top()
35:                    ExpandState(s)
36:                    Insert s in CLOSED_inad
37:            else
38:                if g(s_goal) ≤ OPEN_0.Minkey() then
39:                    if g(s_goal) < ∞ then
40:                        terminate and return path pointed by bp(s_goal)
41:                else
42:                    s ← OPEN_0.Top()
43:                    ExpandState(s)
44:                    Insert s in CLOSED_anchor
```

3.2. Shared Multi-Heuristic A (SMHA*)*

The primary difference between SMHA* and IMHA* is that in SMHA*, the current path for a given state is shared among all the searches, i.e., if a better path to a state is discovered by any of the searches, the information is updated in all the priority queues. As the paths are shared, SMHA* uses a single g (and bp) value for each state, unlike IMHA* in which every search maintains its own g value. Furthermore, path sharing allows SMHA* to expand each state at most *twice*, in contrast to IMHA* which may expand a state up to $n+1$ times (once in each search), and yet achieve the same bounds as IMHA*. We include the pseudocode for SMHA* in Algorithm 2. At this stage, the reader may ignore the mention of $v(.)$ in lines 5, 9 and 21 of the pseudocode presented (Algorithm 2), these lines are added for simplifying the proofs (discussed later) and they do not impact the working of the algorithm.

The Key function and initialization part in SMHA* is the same as in IMHA* other than the fact that SMHA* uses a single g (and bp) variable. Also, IMHA* uses separate CLOSED lists for each search, in contrast SMHA* uses two CLOSED lists, one for the anchor search (CLOSED$_{anchor}$) and another for all the inadmissible searches (CLOSED$_{inad}$). After the initialization, SMHA* runs the inadmissible searches in a round robin manner as long as the check in line 29 is satisfied. If the check is violated for a given search, it is suspended and a state is expanded from OPEN$_0$.

The key difference between SMHA* and IMHA* lies in the state expansion method (ExpandState routine). In SMHA*, when a state s is expanded, its children ($s' \in$ Succ(s)) are simultaneously updated in all the priority queues, if s' has not yet been expanded (lines 15-17). If s' has been expanded in any of the inadmissible searches ($s' \in$ CLOSED$_{inad}$) but not in the anchor search (i.e., $s' \notin$ CLOSED$_{anchor}$, check at line 12), it is inserted only in OPEN$_0$. A state s' that has been expanded in the anchor search ($s' \in$ CLOSED$_{anchor}$) is never re-expanded and thus, never put back into any of the priority queues.

The only exception to this simultaneous update (for a state s' not yet expanded) is the optimization at line 16 which ensures that s' is not put into OPEN$_i$ if Key$(s', i) > w_2 *$ Key$(s', 0)$, because such a state will never be expanded from OPEN$_i$ anyway (check at line 29). The ExpandState routine also removes s from all OPEN$_i$ (line 4) making sure that it is never re-expanded again in any inadmissible search and not re-expanded in the anchor search if its g is not lowered.

If $g(s_{goal})$ becomes the minimum key value in any of the searches (anchor or inadmissible), SMHA* terminates with a solution within the $w_1 * w_2$ bound that can be obtained by greedily following the bp pointers from s_{goal} to s_{start}. Otherwise, no finite cost solution exists.

Next, we discuss the analytical properties of SMHA*. First, we should note that, unlike IMHA*, the anchor search in SMHA* is not a direct replica of a single shot WA* with a consistent heuristic function and, therefore, we cannot directly use the results for WA* (Theorem 1 and 2) to derive SMHA* properties. Instead, we follow a three step approach to prove the properties of SMHA*. We start by stating some low level properties, in the next phase we use these properties to obtain some key results for the anchor search. Finally, we use those theorems to prove the correctness, bounded sub-optimality and complexity results.

To simplify the proofs, we augment the pseudocode for SMHA* with lines 5, 9 and 21, which show that every state s now maintains an additional variable $v(s)$, which is initially set to ∞, and then is set to the $g(s)$, when s is expanded in any of the searches. It should be noted that this modification does not impact the working of SMHA* in any way as the v values are not used in the algorithm. We also extend the initialization assumption used for the IMHA* proofs to include v value, we assume that any state s with undefined v, g and bp values (i.e., s not generated by the search) has $v(s) = g(s) = \infty$ and $bp(s) = \textbf{null}$.

Lemma 1. *At any point of time during the execution of SMHA* and for any state s, we have* $v(s) \geq g(s)$.

Proof. The lemma clearly holds before line 27 since at that point all the v values are infinite. Afterwards, g values can only decrease (line 11). For any state s, on the other hand, $v(s)$ is initiated to ∞ (line 9) and only changes on line 5 when it is set to $g(s)$. Thus, it is always true that $v(s) \geq g(s)$. □

Lemma 2. *At line 27 (and line 28), all v and g values are non-negative, $bp(s_{start}) = \textbf{null}$, $g(s_{start}) = 0$ and for $\forall s \neq s_{start}, bp(s) = argmin_{(s' \in Pred(s))} v(s') + c(s', s),\ g(s) = v(bp(s)) + c(bp(s), s)$.*

Proof. The lemma holds after the initialization, when $g(s_{start}) = 0$ while the rest of g values are ∞, and all the v values are ∞.

The only places where g and v values are changed afterwards are on lines 11 and 5. If $v(s)$ is changed in line 5, then it is decreased according to Lemma 1. Thus, it may only decrease the g values of its successors. The test on line 10 checks this and updates the g and bp values if necessary.

Since all costs are non-negative and they never change, $g(s_{start})$ can never be changed: it will never pass the test on line 10, and thus is always 0. The Lemma thus holds. □

Lemma 3. *Suppose s is selected for expansion on lines 34 or 42. Then the next time line 28 is executed $v(s) = g(s)$, where $g(s)$ before and after the expansion of s is the same.*

Proof. Suppose s is selected for expansion. Then on line 5, $v(s) = g(s)$, and it is the only place where a v value changes. We, thus, only need to show that $g(s)$ does not change. It could only change if $s \in \text{Succ}(s)$ and $g(s) > v(s) + c(s, s)$. The second test, however, implies that $c(s, s) < 0$ since we have just set $v(s) = g(s)$. This contradicts our assumption that costs are non-negative. □

Next, we analyze the properties of the anchor search in SMHA* and show that it essentially follows the same lower bound properties as IMHA* (Theorems 1 and 2). At an intuitive level, we can see that the lower bound results for SMHA* should be equivalent with IMHA*, as at any given point the OPEN_0 in SMHA* can be viewed as a superset of OPEN_0 in IMHA* (or WA* without re-expansions). This is due to the fact that whenever a state s is expanded in any of the searches of SMHA* its children are put into OPEN_0, thus it includes states from different searches. On the other hand, although s is deleted from OPEN_0 at this point (line 4), it can be re-inserted later if a better path to it is discovered (lowered g value), as long as it has not yet been expanded in the anchor search. Now, as both Theorem 1 and 2 refer to the minimum key value in OPEN_0, the same bounds should hold for the anchor search in SMHA*.

However, to prove these results formally, we consider a set of states defined as follows,

Definition 1.

$$Q = \{u | v(u) > g(u) \bigwedge v(u) > w_1 * g^*(u)\} \tag{6}$$

In other words, Q contains all the states that are generated but not yet expanded ($v(u) \neq g(u)$) and has $v(u)$ more than w_1 times the best cost path between s_{start} and u.

Theorem 6. *At line 27 (and line 28), let Q be defined according to the Definition 1. Then for any state s with* $\text{Key}(s, 0) \leq \text{Key}(u, 0)\ \forall u \in Q$, *it holds that $g(s) \leq w_1 * g^*(s)$.*

Proof. We prove by contradiction.

Suppose there exists an s such that $\text{Key}(s, 0) \leq \text{Key}(u, 0) \forall u \in Q$, but $g(s) > w_1 * g^*(s)$. The latter implies that $g^*(s) < \infty$. We also assume that $s \neq s_{start}$ since otherwise $g(s) = 0 = g^*(s)$ (Lemma 2).

Let us consider a least cost path from s_{start} to s given as $P = \Pi(s_0 = s_{start}, ..., s_k = s)$. The cost of this path is $g^*(s)$. Such path must exist since $g^*(s) < \infty$. Our assumption that $g(s) > w_1 * g^*(s)$ means that there exists at least one

$s_i \in \Pi(s_0, ..., s_{k-1})$ whose $v(s_i) > w_1 * g^*(s_i)$. Otherwise,

$$\begin{aligned} g(s) = g(s_k) &\leq v(s_{k-1}) + c(s_{k-1}, s_k) \text{ (Lemma 2)} \\ &\leq w_1 * g^*(s_{k-1}) + c(s_{k-1}, s_k) \text{ (As } v(s_i) \leq w_1 * g^*(s_i) \; \forall s_i \in P) \\ &\leq w_1 * (g^*(s_{k-1}) + c(s_{k-1}, s_k)) \text{ (As } s_{k-1} \in \text{optimal path to } s_k) \\ &\leq w_1 * g^*(s_k) = w_1 * g^*(s) \end{aligned} \tag{7}$$

Let us now consider $s_i \in \Pi(s_0, ..., s_{k-1})$ with the smallest index $i \geq 0$ (that is, the closest to s_{start}) such that $v(s_i) > w_1 * g^*(s_i)$. We will now show that $s_i \in Q$.

If $i = 0$ then $g(s_i) = g(s_{start}) = g^*(s_{start}) = 0$ (Lemma 2). Thus, $v(s_i) > w_1 * g^*(si) = 0 = g(s_i)$, and $s_i \in Q$. If $i > 0$, we have $v(s_i) > w_1 * g^*(s_i)$, and by the choice of s_i,

$$\begin{aligned} g(s_i) &= min_{(s' \in Pred(s_i))} v(s') + c(s', s) \\ &\leq v(s_{i-1}) + c(s_{i-1}, s_i) \\ &\leq w_1 * g^*(s_{i-1}) + c(s_{i-1}, s_i) \\ &\leq w_1 * g^*(s_i) \end{aligned} \tag{8}$$

We thus have $v(s_i) > w_1 * g^*(s_i) \geq g(s_i)$, which also implies that $s_i \in Q$.

We will now show that Key$(s, 0) >$ Key$(s_i, 0)$, and finally arrive at a contradiction. According to our assumption,

$$\begin{aligned} \text{Key}(s, 0) &= g(s) + w_1 * h_0(s) \\ &> w_1 * g^*(s) + w_1 * h_0(s) \\ &> w_1 * (g^*(s_i) + c^*(s_i, s) + h_0(s)) \\ &> w_1 * g^*(s_i) + w_1 * h_0(s_i) \text{ (h_0 is consistent)} \\ &> g(s_i) + w_1 * h_0(s_i) \\ &> \text{Key}(s_i, 0) \end{aligned} \tag{9}$$

Now, as $s_i \in Q$ and Key$(s_i, 0) <$ Key$(s, 0)$, we have a contradiction to our assumption that Key$(s, 0) \leq$ Key$(u, 0), \forall u \in Q$. □

Theorem 7. *At line 27 (as well as at line 28), let Q be defined according to the Definition 1. Then $Q \subseteq$ OPEN$_0$.*

Proof. We prove this by induction. At the very start OPEN$_0$ contains s_{start}, $v(s_{start}) = \infty$ and $g(s_{start}) = 0$, thus, $s_{start} \in Q$, obviously $s_{start} \in$ OPEN$_0$ (line 24). Also, for all other states s', $g(s') = v(s') = \infty$, thus the statement holds.

Now, we consider the states that are part of CLOSED$_{anchor}$. CLOSED$_{anchor}$ holds all the states that are expanded in the anchor search at line 43 (insertion in line 44). Such a state s is removed from OPEN$_0$ (and from all other OPEN$_i$, $i \neq 0$) at line 4 (before it is inserted in CLOSED$_{anchor}$). It follows that $s \in$ CLOSED$_{anchor} \implies s \notin$ OPEN$_i$, $\forall i,\ i = 0, 1, \ldots, n$, as a state s expanded in the anchor search will never be put back in any open list due to the check at line 12.

First, we show that the following statement, denoted by (*), holds for the first time line 27 is executed.

(*): for any state $s \in$ CLOSED$_{anchor}$, $v(s) \leq w_1 * g^*(s)$.

This is obviously true as before the first execution of line 27, no state is expanded in the anchor search and thus CLOSED$_{anchor}$ is empty. We will now show by induction that the statement and the theorem continues to hold for the consecutive executions of the line 27. Suppose the theorem and the statement (*) held during all the previous executions of line 27. We need to show that the theorem holds the next time line 27 is executed.

We first prove that the statement (*) still holds during the next execution of line 27. Let us consider the for loop in line 28. From the induction hypothesis, we obtain that the theorem and the statement (*) is true before the first iteration of the for loop (as there are no other statements between line 27 and 28). We assume the theorem and statement (*) are true

for all for loop iterations prior to current iteration and show that they will remain true after the current iteration. Therefore, the statements will also remain true when the for loop ends and the control returns to line 27.

Considering the current iteration of the for loop, we observe the following possibilities, either the search terminates, in which case the theorem is proved trivially (as there will no more execution of line 27), or a state s is expanded either in an inadmissible search, i.e., from OPEN$_i$ ($i = 1, 2, \ldots, n$) or in the anchor search, i.e., from OPEN$_0$.

Let us consider the first case, i.e., when s is selected for expansion in OPEN$_i$ (line 34) and $i \neq 0$. This expansion (of s) will only change $v(s)$ (line 5) and no other state's v will be altered. Also, as s is being expanded in an inadmissible search, it follows that $s \notin$ CLOSED$_{anchor}$ (as shown before) and thus there will be no change in the v-values of states that are in CLOSED$_{anchor}$. Thus, all the states that are in CLOSED$_{anchor}$, satisfy $v(s) \leq w_1 * g^*(s)$ before and after such an expansion (from the induction hypothesis). Thus, the statement (*) holds.

Now, consider the case when s is chosen for expansion in the anchor search (line 42). From the induction hypothesis $Q \subseteq$ OPEN$_0$, thus, selection of s guarantees $g(s) \leq w_1 * g^*(s)$ (Theorem 6). From Lemma 3, it also follows that next time line 28 is executed $v(s) \leq w_1 * g^*(s)$, and hence the statement (*) still holds as no other state's v value has changed during this expansion.

We now prove that after s is expanded (either in the anchor or in an inadmissible search) the theorem itself also holds. We assume that $Q \subseteq$ OPEN$_0$ held during all earlier iteration of the for loop and we prove it by showing that Q continues to be a subset of OPEN$_0$ after the current iteration. This will also prove that $Q \subseteq$ OPEN$_0$, when the next time line 27 is executed.

Any state that is generated for the first time in the ExpandState routine will be initialized with $v(s) = g(s) = \infty$. Now, if during this expansion its g value is lowered then $v(s) = \infty > g(s)$. Similarly, for all other states if the g value is lowered (at line 11), it would ensure $v(s) > g(s)$, as either $v(s) = \infty$ or $v(s)$ holds the value $g(s)$ when it was expanded last, now the g value has decreased (the g value of a state can only be altered at line 11, and this change can only lower the g value as otherwise, the check at line 10 will never be true).

Each such state will be inserted/updated in OPEN$_0$ at line 13, the only exception being if the check at line 12 is satisfied, i.e., s has earlier been expanded in the anchor search, in which case $s \in$ CLOSED$_{anchor}$, and thus has $v(s) \leq w_1 * g^*(s)$, from statement (*).

All the states that have $v(s) > g(s)$ are either $\in$ OPEN$_0$ or $\in$ CLOSED$_{anchor}$. From statement (*) any state $s \in$ CLOSED$_{anchor}$ satisfies $v(s) \leq w_1 * g^*(s)$. Therefore, for any state s, $v(s) > g(s)$ and $v(s) > w_1 * g^*(s)$, imply that $s \in$ OPEN$_0$. Which means, $Q \subseteq$ OPEN$_0$.

Thus proved. □

Theorem 8. *At line 28, for any state s with* Key$(s, 0) \leq$ Key$(u, 0)$ $\forall u \in$ *OPEN$_0$, it holds that* $g(s) \leq w_1 * g^*(s)$.

Proof. This theorem can be proved directly from the results of Theorem 6 and 7. From Theorem 7 we have $Q \subseteq$ OPEN$_0$, thus any state that satisfies Key$(s, 0) \leq$ Key$(u, 0)$, $\forall u \in$ OPEN$_0$ also satisfies Key$(s, 0) \leq$ Key$(u, 0)$ $\forall u \in Q$. Thus, from Theorem 6, $g(s) \leq w_1 * g^*(s)$. □

Theorem 9. *At line 28, it holds that OPEN$_0$.*Minkey$() \leq w_1 * g^*(s_{goal})$.

Proof. We can prove this in same manner as done for Theorem 2, utilizing Theorem 8 instead of Theorem 1. □

Theorem 10. *When SMHA* exits, $g(s_{goal}) \leq w_1 * w_2 * g^*(s_{goal})$, i.e., the solution cost is bounded by $w_1 * w_2$ sub-optimality factor.*

Proof. This theorem can be proved in a manner similar to the proof for Theorem 3 using Theorems 8 and 9. □

Theorem 11. *During the execution of SMHA*, a) no state is expanded more than twice, b) a state expanded in the anchor search is never re-expanded, and c) a state expanded in an inadmissible search can only be re-expanded in the anchor search if its g value is lowered.*

Proof. In SMHA*, a state s can only be expanded when it is selected as the top state of OPEN$_i$ in either line 34 or 42.

If s is selected for expansion in line 34, the very next call is to the function ExpandState (line 35), which removes this selected state from OPEN$_i, \forall i = 0..n$ (line 4). Also, after the ExpandState call, s is inserted in CLOSED$_{inad}$ (line 36). Now, a state (other than s_{start}) can only be inserted in OPEN$_i$ ($i \neq 0$) in line 17. If a state s has already been expanded in any of the inadmissible searches (i.e., $s \in$ CLOSED$_{inad}$), the check at line 14 will ensure that s is not inserted again in OPEN$_i$ ($i \neq 0$).Therefore, a state can only be expanded once in the inadmissible searches.

Now, when a state s is expanded in the anchor search, similar to the earlier case, here also, s is removed from all OPEN$_i$ (line 4) and inserted to CLOSED$_{anchor}$. Thus, s can only be expanded again either in inadmissible searches or in anchor search, if it is re-inserted in any of the OPEN$_i$, which can only be done in lines 13 or 17. However, as $s \in$ CLOSED$_{anchor}$, the check at line 12 will never be true, thus the control will never reach lines 13 or 17, i.e., s will never be re-expanded. Therefore, statement b) is true. Also, as s can be expanded at most once in the anchor search and at most once in the inadmissible searches, i.e., s cannot be expanded more than twice, proving statement a).

Finally, a state s that has been expanded in an inadmissible search, can only be expanded in the anchor search later if it is re-inserted in OPEN$_0$. A state can only be inserted in OPEN$_0$ (any OPEN$_i$, for that matter) if the check at line 10 is true, i.e., if its g value is less than its earlier g value. Thus, a state s whose g has not been lowered after its expansion in any inadmissible search will never satisfy the condition at line 10 and will not be re-inserted in OPEN$_0$ and thus, can never be expanded in the anchor search. Therefore, statement c) is true. □

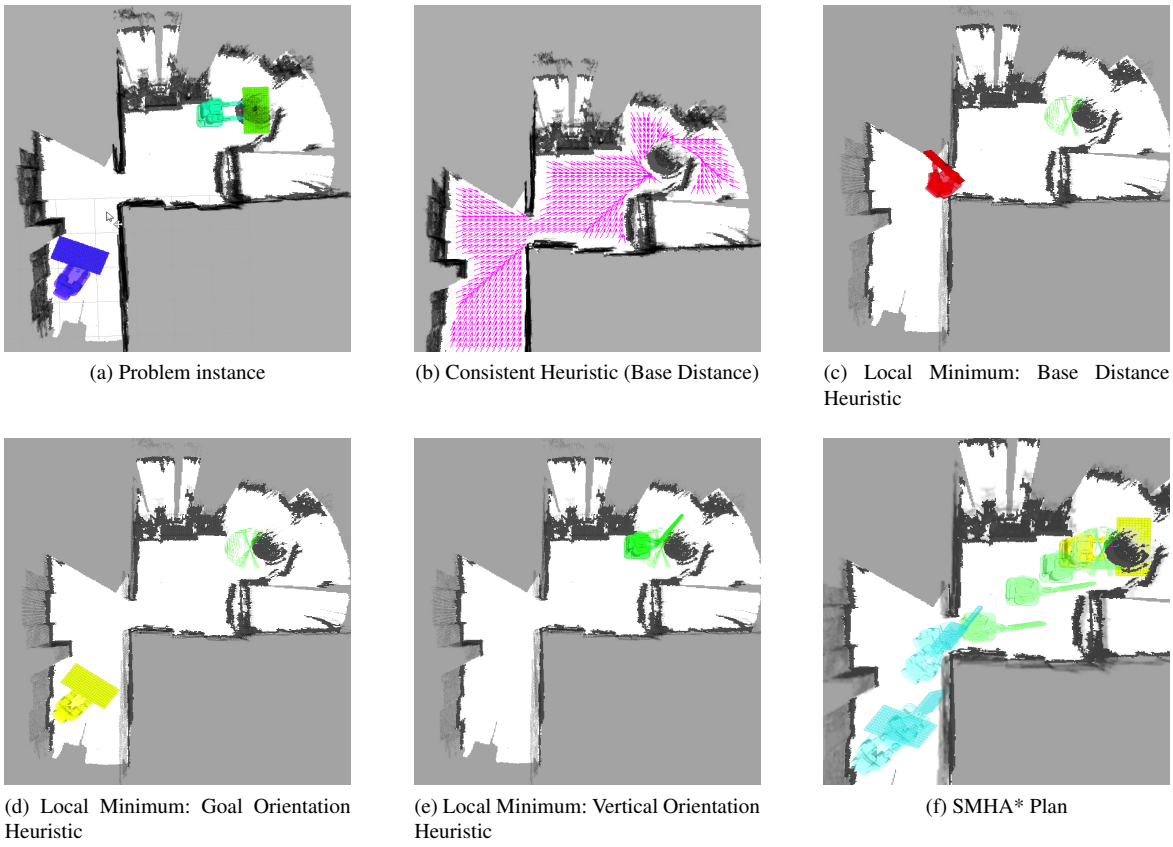
Theorem 12. *In SMHA*, a state s is never expanded in the i^{th} inadmissible search if* Key$(s, i) > w_1 * w_2 * g^*(s_{goal})$.

Proof. The proof is similar to Theorem 5, utilizing the fact that OPEN$_0$.Minkey$() \leq w_1 * g^*(s_{goal})$ (Theorem 9). □

Theorem 10 shows that SMHA* guarantees the same sub-optimality bounds as IMHA* while Theorem 11 highlights the difference in complexity between these two approaches. In IMHA*, a state can be re-expanded at most $n + 1$ times as each search is performed independently, whereas in SMHA* the same bounds are attained with at most 1 re-expansion per state (at most one expansion in inadmissible searches and one expansion the anchor search, when the state is first expanded in an inadmissible search and its g value is lowered). On the other hand, IMHA* has the following advantages over SMHA*, a) expansion of states in IMHA* is cheaper than SMHA* as SMHA* may require $n + 1$ insertion/update/removal steps, whereas IMHA* requires only 1, b) in SMHA* all the searches store a copy of each of the generated states, thus the memory overhead is more[4], and c) IMHA* is more amenable to parallelization, as individual searches do not share information.

A more important distinction between SMHA* and IMHA* arises from the fact that as SMHA* shares the states and the best path information among all the searches[5], it can potentially use a combination of partial paths to exit from depression regions, which is not possible in IMHA*. Therefore, if there are nested depression regions in the state space that none of the consistent/inadmissible heuristics can avoid independently, SMHA* can outperform IMHA*. In Figure 2, we illustrate this phenomenon with an example of a 12D planning problem. As shown in the figure, we use 3 heuristics here (1 consistent + 2 inadmissible), however none of these heuristics can guide the search completely, as all of them get stuck in their own local minimum. In such a scenario, SMHA* can use partial paths obtained by individual searches and seamlessly combine them to obtain a complete plan (Figure 2f), but IMHA* cannot, as it does not share information. A video of the PR2 robot actually performing the task using the plan generated by SMHA* is included in Extension 1.

[4] It may be noted that this memory overhead can be eliminated by using a single open list and making the update/insertion/removal more informed.
[5] SMHA* uses common g and bp values for all the searches. Also during each expansion, the children of the expanded state are inserted/updated in all the priority queues making them available for selection according any of the heuristics.



(a) Problem instance

(b) Consistent Heuristic (Base Distance)

(c) Local Minimum: Base Distance Heuristic

(d) Local Minimum: Goal Orientation Heuristic

(e) Local Minimum: Vertical Orientation Heuristic

(f) SMHA* Plan

Fig. 2. A full-body (12D) planning example highlighting the advantage of SMHA* over WA*/IMHA* for cases where no individual heuristic can escape all the depression regions by itself (nested depression regions). Here the task for the robot (PR2) is to carry a large object (picture frame) through a narrow corridor and a doorway, and then to put it down on a table. The problem instance is shown in 2a with left and right figures depicting the start and end configurations, respectively. Figure 2b shows the vector field for a consistent heuristic (h_0), computed by performing a backward (from goal to start) 2D search for the PR2 base. Figure 2c shows how a search using h_0 gets stuck (at the door) as it cannot orient the end-effector correctly, and thus ends up expanding a large number of states at the shown position without moving toward the goal (deep local minimum) before running out of time (1 minute). To rectify this, we compute two additional (inadmissible) heuristics by including the orientation information for the end-effector, one targeting the goal orientation (h_1) and another targeting a vertical orientation (h_2). Unfortunately, none of these heuristics are powerful enough to take the search to the goal state as they both suffer from their own depression regions (the searches using h_1 and h_2 gets stuck at different positions as shown in Figures 2d and 2e). As all the heuristics (consistent and inadmissible) lead the search to separate depression regions, IMHA* cannot avoid any of them and performs as poorly as WA*. However, as shown in Figure 2f, SMHA* can efficiently compute a plan by using partial paths from different heuristics. It uses the base and vertical orientation heuristic (in parts) go through the corridor and the door, and then switches to the goal orientation heuristic to align the end-effector to the goal.

It should be noted that while sharing paths certainly makes SMHA* more powerful in handling nested minima, it can also be counter-productive at times, especially if the path sharing drags the search to a local minimum that could have been avoided if the searches were kept separate. Figure 3 includes an example of such a scenario with 3 heuristics (1 consistent + 2 inadmissble). In this case, IMHA* terminates earlier than SMHA*, as path sharing leads both the inadmissible searches toward the same local minimum.

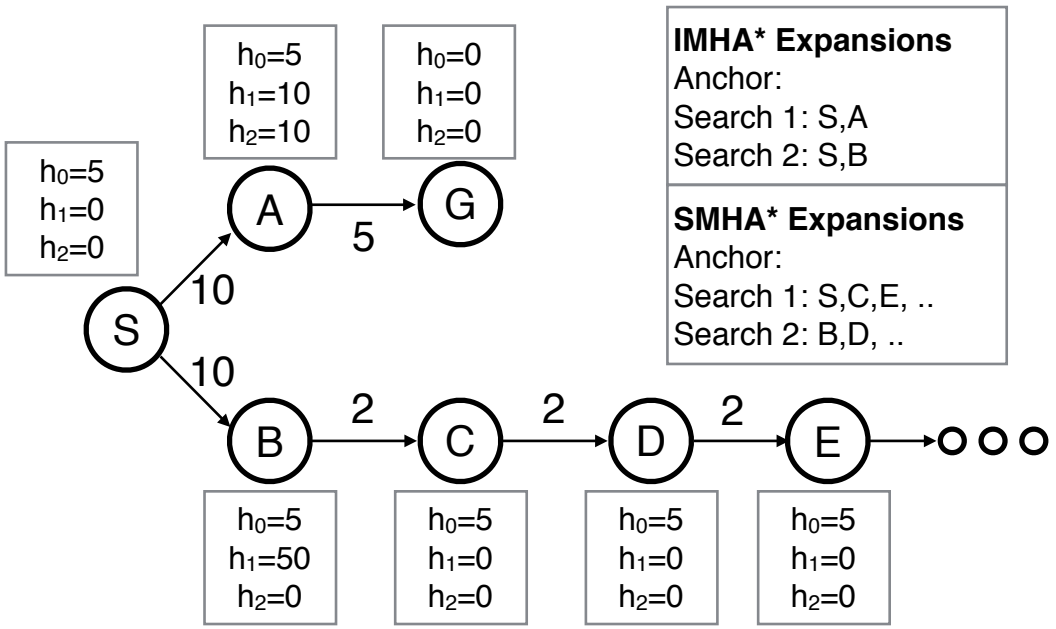


Fig. 3. An example scenario depicting how path sharing in SMHA* can at times degrade performance (over IMHA*). **S** denotes the start state and **G** denotes the goal state. The different h-values are shown in the boxes alongside the states. We run MHA* with $w_1 = 5.0$ and $w_2 = 2.0$. IMHA* does not share information among the searches, which in this case results in termination after 4 expansions (as shown in the figure). In contrast, SMHA* shares the g-values (and the states), which in this case drags the search toward the lower part of the graph (expansions shown in the figure) and thus, delays termination.

4. Experimental Results

We evaluated the performance of MHA* (both IMHA* and SMHA*) for the following domains: 12D mobile manipulation planning for the PR2 robot, 3D (x, y, orientation) navigation for single and multiple goal domains, and sliding tile puzzles. All the experiments were performed on an Intel $i7 - 3770$ ($3.40GHz$) PC with $16GB$ RAM.

We primarily benchmarked MHA* against WA* without re-expansions (as in ARA* (Likhachev et al., 2004)/RWA* (Richter et al., 2010)). In addition, we also compared with the sampling-based planning algorithms (PRM (Kavraki et al., 1996), RRT-Connect (Jr. and LaValle, 2000), and RRT* (Karaman and Frazzoli, 2010)), multiple heuristic greedy best first search (MHGBFS) (Röger and Helmert, 2010), multiple parameter WA* (MPWA*) (Valenzano et al., 2010) and Explicit Estimation Search (EES) (Thayer and Ruml, 2011) when applicable.

As MHA* uses two sub-optimality bounds (w_1 and w_2) in comparison to one w used by WA*/MPWA*/EES, we set $w_2 = min(2.0, \sqrt{w})$ and $w_1 = w/w_2$, for all our experiments (and the examples), so that the solutions are guaranteed to be within w-sub-optimality bound.

For MHGBFS, we used the same heuristics as used for SMHA*. For MPWA*, we used the admissible heuristic with 5 different weights ($0.2 \times w$ to $1.0 \times w$, with 0.2 gap; where $w \geq 10.0$). For EES, we used an inadmissible distance measure, one inadmissible heuristic function (from the set used for MHA*) and the admissible heuristic. We ran WA*, MPWA* and MHGBFS without state re-expansions as re-expansions can degrade the planning time. Both WA* and MPWA* can satisfy the quality bounds without re-expansions, while MHGBFS does not guarantee any bounds.

4.1. Mobile Manipulation Planning for the PR2 (12D)

The PR2 mobile manipulation robot is a dual-arm robot (7 degrees of freedom each) with an omni-directional base and a prismatic spine. In our experiments, we used a state space representation similar to that used in (Cohen et al., 2012). We represented a robot state with 12 degrees of freedom: a 6 DOF object pose, 2 redundant arm joints, 2D Cartesian

coordinates for the base, an orientation of the base, and the prismatic spine height. The planner was provided the initial configuration of the robot as the start state. The goal state contained only the 6 DOF position of the object, which made it inherently under-specified because it provides no constraints on the position of the robot base or the redundant joint angles. The actions used to generate successors for states were a set of motion primitives, which are small, kinematically feasible motion sequences that move the object in 3D Cartesian space, rotate the redundant joint, or move the base in a typical lattice-type manner (Likhachev and Ferguson, 2009). The prismatic spine was also allowed to adjust its height in small increments.

We computed the admissible heuristic by taking the maximum value between the end-effector heuristic and the base heuristic, where the end-effector heuristic was obtained by a 3D Dijkstra search initialized with the (x,y,z) coordinates of the goal and with all workspace obstacles inflated by their inner radius, and the base heuristic was obtained using a 2D Dijkstra search for the robot base where the goal region is defined by a circular region centered around the (x,y) location of the 6 DOF goal. The purpose of this circular region is to maintain an admissible heuristic despite having an incomplete search goal. As the set of possible goal states must have the robot base within arm's reach of the goal, we ensure that the heuristic always underestimates the actual cost to goal by setting the radius of the circular region to be slightly larger[6] distance than the maximum reach of the robot arm.

For IMHA*, we computed 2 additional heuristics in the following way. First, we randomly selected 2 points from the base circle around the goal with valid inverse kinematic (IK) solutions for the arm to reach the goal and ran 2D Dijkstra backward searches (to the start state) starting from these 2 points. This gave us two different base distances. Second, we computed an orientation distance by obtaining the angular difference between the current base orientation (at a given point) and the desired orientation, which was to make the robot face the end-effector goal. These distances (base and orientation) were then added to the end-effector heuristic to compute the final heuristic values. Note that this informative heuristic is clearly inadmissible, as they compute the angular and base distance difference for particular configurations, in contrast to finding the minimum value among all possible configurations, but can still be used in the MHA* framework.

For SMHA*, we augmented this set by using the base (2D Dijkstra + orientation) and the end-effector heuristics (3D Dijkstra) as two additional heuristics, since SMHA* can share the paths among the inadmissible searches, and hence, can potentially benefit from not combining the two heuristics into a single one.

In Figure 4, we include an example of the test scenario with two large tables and a few narrow passageways. For each trial of the experiment, we randomly generated a full robot configuration anywhere in the kitchen for the start state, while generating a valid goal state that lies above the tabletops containing clutter. We generated 15 such environments by randomly changing the object positions and for each such environment we used 10 different start and goal configurations.

	WA*	MHGBFS	MPWA*	EES	IMHA*	SMHA*
SR	31%	76%	36%	27%	70%	81%
SE	1.08	0.78	3.84	1.54	1.58	1.0
RT	0.99	0.91	2.82	1.54	1.41	1.0
SC	0.95	1.57	0.97	0.93	1.09	1.0

Table 1: Comparison between WA*, MHGBFS, MPWA*, EES and MHA* for PR2 manipulation planning in kitchen environments. The first row (SR) shows the percentage of total problem instances solved by each planner. The other rows include the results as a ratio between the algorithm marked in the column heading and the corresponding SMHA* numbers, when both of them solved an instance. Legend: SR - success rate, SE - state expansion ratio, RT - runtime ratio, SC - solution cost ratio.

In Table 1, we include the results comparing WA*, EES, MPWA*, MHGBFS with the MHA*. We used $w = 50$ for all the algorithms. Each planner was given a maximum of 60 seconds to compute a plan. The results clearly show

[6] We use a slightly larger distance to be conservative and thereby avoid any discretization errors which may make the heuristic inadmissible. However, if perfect measurements are available, the exact value of the maximum reach of the robot arm can be used as the radius.

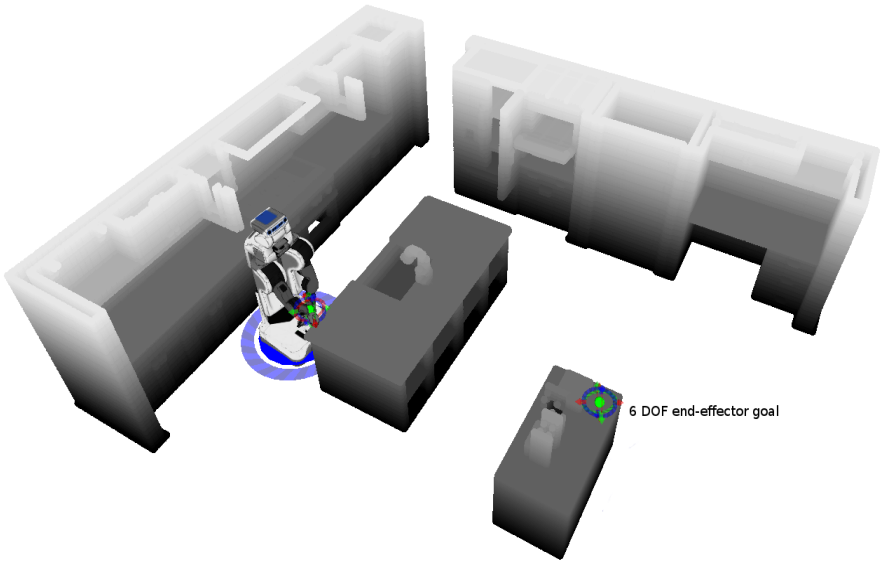


Fig. 4. An example of the planning scenarios used for 12D mobile manipulation. Experiments were done for a kitchen-like environment with narrow passages, tables and shelves. This figure shows a typical problem instance with a fully specified start configuration and a partially specified (6 DOF) goal configuration.

that MHA* (especially SMHA*) and MHGBFS perform much better than WA*/MPWA*/EES, highlighting the efficacy of using multiple heuristics over a single heuristic function, which often suffers from local minima due to the robot's orientation, presence of obstacles, etc.

MPWA* performs slightly better than WA* indicating that the size of a local minimum can depend on the weights used. However, it still gets stuck in most of the cases, since it uses the same heuristic (albeit with different weights) for each search. EES performs poorly when the inadmissible distance function has a large depression. Also, the inadmissible and admissible searches in EES do not use weighted heuristics and thus, often get trapped in some cost plateau.

MHA* (and MHGBFS) is less prone to suffer from heuristic depression regions as they can converge in time if any of the heuristics can lead the search to the goal. SMHA* and MHGBFS perform better than IMHA*, as they can use partial paths. For example, they can combine a path obtained in the base heuristic search with the end-effector heuristic search. MHGBFS performs comparably to SMHA* in terms of number of instances solved and slightly better in terms of convergence time. However, the solution costs obtained for MHGBFS are significantly worse than SMHA* (and IMHA*), as noted in Solution Cost ratio in Table 1. This highlights the utility of the anchor search, which ensures better quality solution by intelligently controlling the inadmissible expansions.

	PRM	RRT-Connect	RRT*(First)	RRT*(Final)	IMHA*	SMHA*
SR	74%	98%	100%	100%	70%	81%
RT	2.07	0.18	5.39	8.48	1.41	1.00
BD	1.93	1.88	1.36	1.34	1.02	1.00
ED	1.87	1.68	1.27	1.24	0.99	1.00

Table 2: Comparison between MHA* and sampling-based planners for PR2 manipulation in kitchen environments. All the results are presented as a ratio between the algorithm marked in the column heading and the corresponding SMHA* numbers. For sampling-based planners, the distances are obtained after post processing. Since RRT* is an anytime algorithm, we include the results for the first solution reported (RRT*-First) and the solution obtained at the end of 60 secs (RRT*-Final). Legend: SR - success rate, RT - runtime ratio, BD - base distance ratio, ED - end-effector distance ratio.

In Table 2, we include the results comparing MHA* with 3 sampling based algorithms, namely PRM, RRT-Connect and RRT*, in terms of runtime and solution quality. For the sampling-based algorithms we used the standard OMPL (Şucan et al., 2012) implementation. Since the sampling-based planners do not directly report the solution costs, in this table we include the results in terms of base and end-effector distances covered by the robots (after post processing). All the results are presented as a ratio over the corresponding SMHA* numbers for episodes where that planner and SMHA* were both successful in finding a solution within 60 seconds).

The results show that SMHA* performs reasonably well when compared to the sampling-based planners. Its runtime is better than both PRM (5X) and RRT* (8X) but worse than RRT-Connect (5X). In terms of solution quality, MHA* results are noticeably better than all the sampling-based planners. However, both RRT-Connect and RRT* can solve more number of instances, mainly due to the facts that a) they are not bound by discretization choices and b) they do not use any heuristic function that may lead to local minima. Overall, the results show that MHA* (SMHA* in particular) is a reasonable alternative for planning in such high-dimensional environments, especially when we are looking for predictable[7] and bounded sub-optimal (with respect to the discretization choice) planning solutions.

4.2. 3D Path Planning (navigation)

While high dimensional problems like full-body planning for the PR2 are a true test-bed for assessing the real life applicability of MHA*, finding close-to-optimal solutions in such spaces is infeasible. Therefore, in order to get a better idea of MHA*'s behavior for close-to-optimal bounds, we ran experiments in an easier 3D (x, y, orientation) navigation domain.

Here, we modeled our environment as a planar world and a rectangular polygonal robot with three degrees of freedom: x, y, and θ (heading). The search objective is to plan paths that satisfy the constraints on the minimum turning radius. The actions used to get successors for states are a set of motion primitives used in a lattice-type planner (Likhachev and Ferguson, 2009). We computed the consistent heuristics (h_0) by running a 16-connected 2D Dijkstra search assuming the robot is circular with a radius equal to the actual robot's (PR2 base) inscribed circle, i.e., inflating the objects by the robot's in-radius.

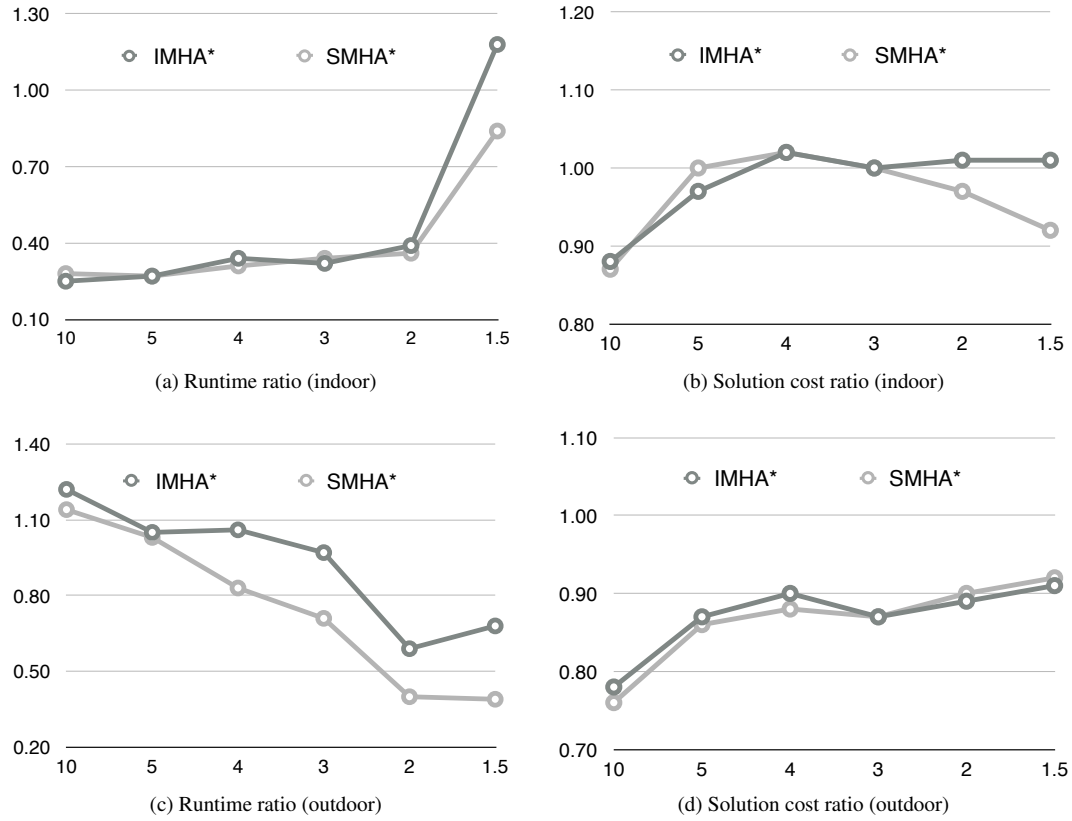
We used two kinds of environments for the testing: i) simulated indoor environments, which are composed of a series of randomly placed narrow hallways and large rooms with polygonal obstacles, and ii) simulated outdoor environments, which have relatively large open spaces with random regular shaped obstacles[8], that occupy roughly $10-30\%$ of the space. We generated 100 maps of 1000×1000 dimensions for both these environments. We performed the following experiments.

3D path planning with dual heuristics: In this experiment, in addition to h_0, we generated an extra heuristic h_1 by performing another 2D Dijkstra search by inflating all the objects using the robot's out-radius. Obviously, this heuristic function is an arbitrarily inadmissible one as a valid path can get blocked by such an inflation. On the other hand, h_1 is less prone to the local minima created due to minimum turning radius constraints.

In Figure 5, we include the results obtained for this experiment. The results show that for indoor environments, in terms of runtime, MHA* generally outperforms WA* by a significant margin. This is because, in indoor maps, the presence of large rooms and narrow corridors frequently creates big depression regions. MHA* can utilize the out-radius heuristic to quickly get away from such depression zones, while maintaining the quality bounds using the anchor search. In contrast, WA* only uses the in-radius heuristic guidance and thus often gets stuck in large local minima. For low sub-optimality bounds, IMHA*'s performance degrades a bit compared to WA* due to re-expansions, however SMHA* still performs better. Overall, MHA* provides around 3X speed up over WA* while producing solutions of similar quality. In contrast, for outdoor environments, all the algorithms perform similarly for high bounds. In fact, in most cases, MHA* has a trifle

[7] One of the issues with sampling based planners is that they often produce completely different solutions for problems that are very similar Phillips et al. (2012), in contrast, the search based planners tend to produce more predictable solutions.

[8] We used circular, triangular and rectangular obstacles.



(a) Runtime ratio (indoor)

(b) Solution cost ratio (indoor)

(c) Runtime ratio (outdoor)

(d) Solution cost ratio (outdoor)

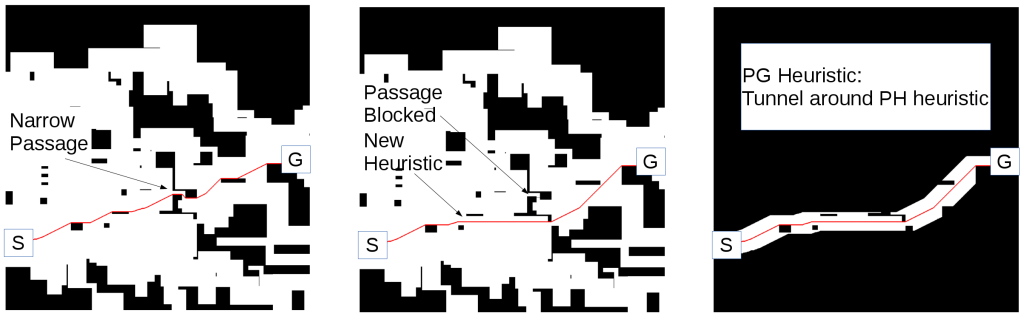
Fig. 5. Comparison between WA* and MHA* with dual heuristic for simulated indoor and outdoor environments in terms of average runtime and solution costs. All the results are shown as a ratio between a candidate algorithm (as mentioned in the plots) and the corresponding WA* numbers. The x-axis in each figure shows the target sub-optimality bounds (w in case of WA*, $w_1 * w_2$ in case of MHA*).

longer runtime than WA* due to overhead of generating an extra heuristic. The comparative results are different for outdoor maps mainly due to the fact that the outdoor maps are generally more benign and large depression regions are rare and thus, WA* is very efficient. However, even in this case, MHA* (especially SMHA*) perform better in case of low bounds, both in terms of solution quality and runtime. This is because MHA* spends more effort in exploring the feasible solutions when compared to WA*.

Comparing IMHA* and SMHA* we notice that there is not much difference in the results, as we used just one extra heuristic, the scope of sharing is minimal. Although, there were instances where SMHA* converged faster, as it could share the search efforts between the anchor and the inadmissible search, which is not possible for IMHA*.

3D path planning with progressive heuristics: In this experiment, we evaluated MHA* on 3D path planning with more than two heuristics. We used two schemes to generate the inadmissible heuristics, Progressive Heuristics (PH) and Progressive Heuristics with Guidance (PG) Holte et al. (1995).

In PH, the 2D Dijkstra path obtained from the original map is investigated for possible bottlenecks. If the path has narrow passages (passage width $\leq$ robot's outer diameter), those are blocked to create the next map, which is then used to compute the next heuristics. The procedure is iterated (at most 10 times), while there are narrow passages. The objective of this strategy is to obtain alternate heuristics until a *relatively wide* path is discovered. In PG, the PH scheme is augmented with an extra heuristic computed from a map with all cells blocked but a tunnel (of width 30 cells) around the last 2D path, with an intention to focus the 3D search toward the *wide* path.

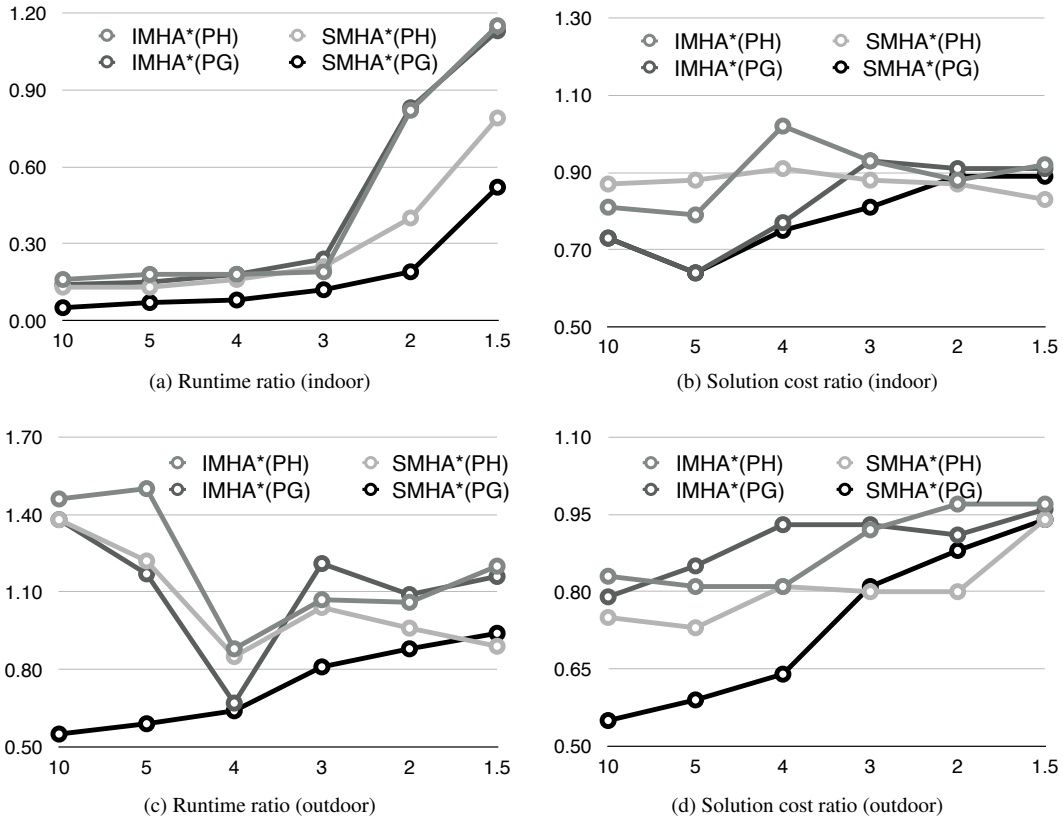


(a) (b) (c)

Fig. 6. Example of progressive heuristics generation. 6a shows the original 2D path with a narrow passage. 6b shows the new map with the second heuristic function (PH), and 6c shows the heuristic created by constructing a tunnel around the path (PG).

Figure 6 depicts an example of the heuristics generation process, where the first 2D path has a narrow passage, which is blocked to obtain the second heuristic(PH). For PG, an extra heuristic is computed from the third map blocking all the cells other than a tunnel around the 2D path.

We used an incremental search (Koenig et al., 2004) to compute the additional heuristics. Typically, the generation of MHA* heuristics (including heuristic computation, path tracing and blocking) took about $0.1 - 0.3$ seconds time, which was around $1.2 - 1.5$ times more than the time it takes to generate a single consistent heuristic for WA*.



(a) Runtime ratio (indoor)

(b) Solution cost ratio (indoor)

(c) Runtime ratio (outdoor)

(d) Solution cost ratio (outdoor)

Fig. 7. Comparison between WA* and MHA* with PH and PG heuristic for simulated indoor and outdoor environments. All the results are shown as a ratio between a candidate algorithm (as mentioned in the plots) and the corresponding WA* numbers. The x-axis in each figure shows the target sub-optimality bounds (w in case of WA*, $w_1 * w_2$ in case of MHA*).

We include the results for this experiment in Figure 7. While the overall trends resemble the observations with dual heuristics (i.e., MHA* provides considerable improvement of WA* for indoor cases), the difference between WA* and MHA* is even more pronounced with PH/PG schemes. For indoor environments, the PH scheme generated additional heuristics for 44 maps (out of 100), whereas for outdoor environments, it generated additional heuristics for 8 maps only. On an average, for indoor environments, MHA* provides 18X-2X runtime improvement over single heuristic WA*. Runtime results for all the algorithms are similar in case of outdoor environments (at high bounds, MHA* is a bit worse). However, for low bound values (≤ 2.0), SMHA*(PG) outperforms WA* both in terms of runtime and solution quality. Comparing IMHA* and SMHA*, we observe that in this experiment SMHA* consistently outperforms IMHA*, highlighting the efficacy of path sharing to navigate around depression regions in a more robust manner. Also, for low bound values, IMHA* tends to re-expand a lot more states which considerably degrades its performance.

	MHGBFS	MPWA*	IMHA*	SMHA*
IS	37	35	52	52
RT	22.39	3.38	1.35	1.00
SC	7.27	1.26	1.22	1.00

Table 3: Comparison between MHGBFS, MPWA*, and MHA* for 3D path planning. All the planners were given maximum 5 secs to plan. IS denotes the number of instances solved. RT and SC denote the runtime and solution cost ratio over SMHA*.

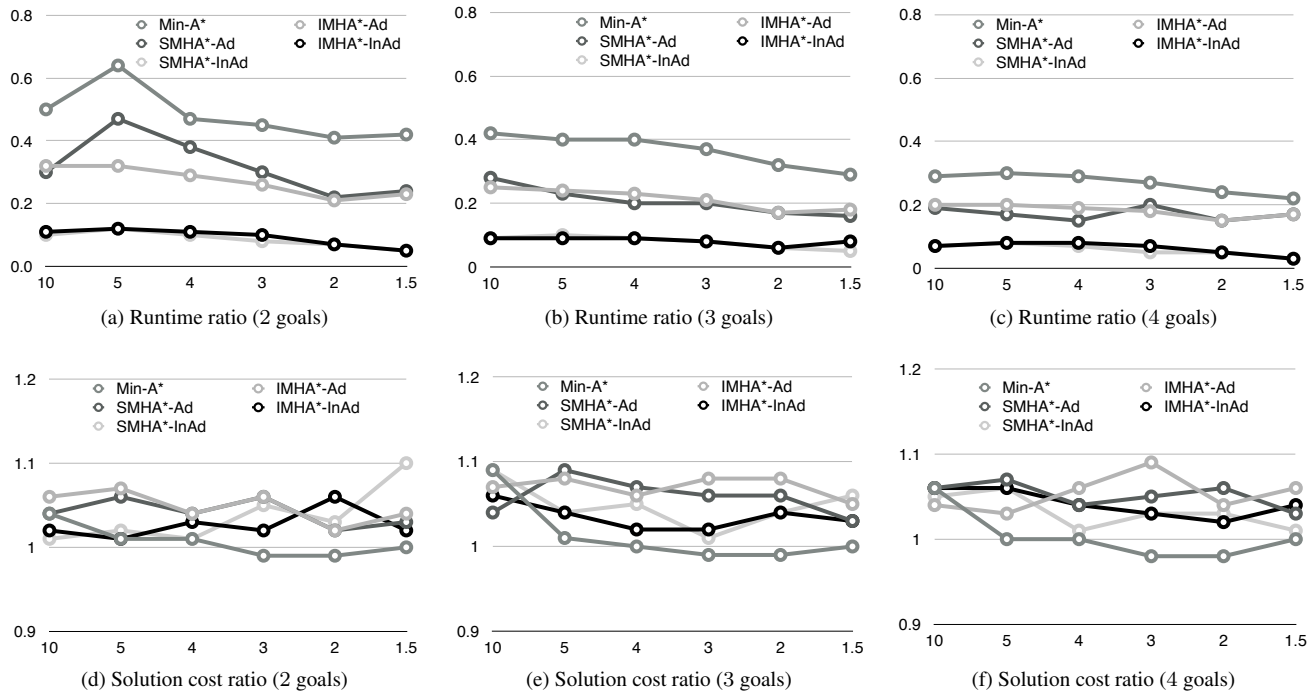
In Table 3, we include the results comparing MHA* (PG) with the MHGBFS and MPWA* on the combined set of 52 *hard* instances (44 indoor + 8 outdoor), for which the PH scheme generated extra heuristics. For MHA* and MPWA*, we used $w = 10.0$. Each algorithm was given a maximum of 5 seconds to compute a plan. Overall, the comparison between MPWA* and MHA* show a similar trend as seen for the 12D planner. In contrast, MHGBFS' performance degrades considerably for this domain as its greedy approach at times drives the search deeper into a local minimum. Unlike 12D planning, cost plateaus are rare here while large depression regions are pretty common, and thus, the greedy approach hurts more often than it helps.

4.3. Multiple Goal Path Planning

In this experiment, we investigate the multiple goal path planning problem for 3D ($x, y, orientation$) lattices. Here, the planning objective is to plan a path to any of the goals from a given set. Such planning problems are common in real life, for example, if a parking lot has multiple open slots, an autonomous car needs to plan a path to any one of the available slots, preferably the one closest to it. MHA* can be a good fit for such planning problems as it can simultaneously explore paths to all the goals, using one (or more) heuristics targeted toward a particular goal.

We used two bounded sub-optimal search techniques to compare against MHA* for this domain. In the first technique (we call it Seq-A*), we sequentially searched for paths to individual goals, starting with the goal that has the minimum heuristic value (2D distance) and using the best cost in the previous searches as an upper bound constraint for the current search, i.e., we prune the states that have f value ($f = g + h$) greater than or equal to the best solution obtained in the earlier iterations. In the second technique (Min-A*), we generated a single admissible heuristic for each state by using the minimum heuristic value across all goals (computed by running a single Dijkstra search considering all goals), and conducted WA* using this heuristic (with inflation).

We ran the MHA* algorithms in two modes, in the first case (MHA*-Ad), we used a consistent heuristic (as computed in Min-A*) for the anchor search and used individual (admissible) heuristics for the inadmissible searches. In the second version (MHA*-InAd) we used the same heuristic for the anchor but used outer-radius based inadmissible heuristics for the other searches. We performed this experiment with indoor environments only (on 100 maps) by randomly selecting $2 - 4$ goals.



(a) Runtime ratio (2 goals) (b) Runtime ratio (3 goals) (c) Runtime ratio (4 goals)

(d) Solution cost ratio (2 goals) (e) Solution cost ratio (3 goals) (f) Solution cost ratio (4 goals)

Fig. 8. Results for multiple goal search for simulated indoor environments. All the results are shown as a ratio between a candidate algorithm (as mentioned in the plots) and the corresponding Seq-A* numbers. The x-axis in each figure shows the target sub-optimality bounds (w in case of Seq-A* and Min-A*, $w_1 * w_2$ in case of MHA*).

The results of this experiment are included in Figure 8, which shows that according to runtime for convergence, MHA*-InAd $<<$ MHA*-Ad $<$ Min-A* $<<$ Seq-A*, and the differences increase with number of goals. Solution quality wise, all the algorithms perform similarly (within 10%). The runtime difference among the algorithms can be explained by the fact that Seq-A* can perform poorly if any of the heuristics suffer from large local minima. Min-A* performs poorly if the *combined* consistent heuristic suffers from local minima. MHA*-Ad can converge faster than Seq-A* or Min-A* if any of the individual goal heuristics does not have a large local minimum, but can get stuck if all the heuristics suffer from local minima (note that for individual goals, the corresponding heuristics are admissible). MHA*-InAd takes maximum advantage of the multi-heuristic approach as it can use out-radius based heuristic to conduct the individual searches (which are less prone to get stuck in a local minimum) and if any one of the inadmissible searches reach a goal state while satisfying the chosen bounds, it can terminate. Overall, the results corroborate our observation that MHA* is indeed well suited to efficiently solve multiple goal search problems and they can provide significant improvement over individual/combined heuristic searches.

4.4. Sliding Tile Puzzle

In this section, we present the experimental results for sliding tile puzzles, a well known discrete optimization problem from AI literature. For this domain, we used the Manhattan distance (MD) plus linear conflicts (LC) ($h_0 = MD + LC$) as the consistent heuristic function (for more details about these heuristics, please see *http://heuristicswiki.wikispaces.com/N++Puzzle*).

For MHA*, we generated 4 additional heuristics by computing the number of misplaced tiles (MT), and adding MD, LC and MT with random weights (between 1.0 to 5.0), i.e., we used $h_i = r_1 * MD + r_2 * LC + r_3 * MT$, where r_1,

r_2, r_3 are random numbers between 1.0 and 5.0. Clearly, these heuristics are somewhat random and are inadmissible. We tested the algorithms on 50 random instances (all solvable) of 48, 63 and 80 puzzles.

		Time limit = 1 minute				Time limit = 3 minutes			
Size	Bound	WA*	IMHA*	SMHA*	WA*(R)	WA*	IMHA*	SMHA*	WA*(R)
48	50	46	49	50	49	47	49	50	50
	20	49	47	50	43	49	49	50	46
	10	45	37	50	32	46	45	50	32
	5	37	19	49	-	38	39	50	-
	2	12	4	9	-	12	6	10	-
63	50	25	35	40	26	29	38	44	33
	20	34	26	39	18	35	37	41	31
	10	32	21	39	17	35	29	40	18
	5	19	8	31	-	24	19	37	-
	2	3	4	4	-	7	4	9	-
80	50	17	24	31	15	22	26	33	23
	20	22	17	27	13	22	27	37	16
	10	19	19	29	12	21	25	30	18
	5	17	11	22	-	20	14	28	-
	2	7	1	4	-	7	1	9	-

Table 4: Number of sliding tile puzzle instances solved by WA*, IMHA* and SMHA* for different sub-optimality bounds with time limit 1 minute (columns 3, 4 and 5) and 3 minutes (columns 7, 8 and 9). WA*(R) columns (6 and 10) show the results obtained by WA* used the same randomized heuristic, as used for MHA*.

In Table 4, we include the results in terms of the number of problems solved by all the algorithms under two time limits: 1 minute and 3 minutes (we did not use larger time bounds due to memory limitations). The results show that even with a randomized approach MHA*, especially SMHA*, can significantly outperform WA*. The performance gap is more pronounced for larger sized problems and higher w values. For example, for 63 puzzle, SMHA* solved 44 instances whereas WA* could only solve 35 instances (IMHA* solved 38), for 80 puzzle, SMHA* solved 37 instances while WA* solved 22 instances only.

To highlight that the MHA* results were not due to the quality of randomized heuristic function, we include the results for WA* with a heuristic function (h_r) generated using the same scheme as in MHA*, i.e., $h_r = r_1 * MD + r_2 * LC + r_3 * MT$, where r_1, r_2, r_3 are random numbers between 1.0 and 5.0 (referred to as WA*(R) in Table 4). As, $h_r \leq 10 * h_0$, we computed the results for sub-optimality bounds ≥ 10 only. From Table 4, we see that the WA*(R) results are actually worse than WA* indicating that the improvements obtained by MHA* are due to their multi-heuristic exploration, which at times helped the search avoid getting trapped in depression regions.

In Table 5, we present the results obtained by comparing MHA* to MHGBFS and MPWA*. MHA*s and MPWA* were run with $w = 20.0$. All the algorithms were given a time limit of 1 minute. From the results, we observe that while MPWA* and MHGBFS perform better in this domain (compared to path planning), overall, SMHA* still remains the best algorithm. For each size, SMHA* could solve more or equal number (as in the case of 48 puzzle) of instances than both MPWA*/MHGBFS. Although MHGBFS had a better planning time in two scenarios, its solution quality is markedly worse, as one would expect from the greedy approach.

Overall, the results from all these domains (from inherently continuous domains such as 12D motion planning to purely discrete domains such as sliding tile puzzles) highlight the potential of MHA* for solving complex optimization problems where it is hard to design a single consistent heuristic which effectively captures all the features of a given problem. Comparing two MHA* strategies, we observe that SMHA* generally dominates IMHA*, as it can combine the advantages of three key MHA* principles, a) simultaneous exploration, b) sharing of partial paths, and c) controlling unnecessary

Size		MHGBFS	MPWA*	IMHA*	SMHA*
48	IS	50	44	38	50
	RT	0.63	2.57	2.63	1.00
	SC	3.59	1.02	0.96	1.00
63	IS	29	27	26	39
	RT	1.06	1.46	1.13	1.00
	SC	4.98	0.97	0.94	1.00
80	IS	21	16	17	27
	RT	0.85	1.17	0.91	1.00
	SC	3.92	0.94	0.99	1.00

Table 5: Comparison between MHGBFS, MPWA*, IMHA* and SMHA* for sliding tile puzzle. The maximum runtime allowed was 1 minute. Legend: IS - number of instances solved (out of 50), RT - runtime ratio, SC - solution cost ratio (over SMHA*).

expansions using the anchor, and can do this with minimal re-expansions (at most two). However, IMHA* has a lower memory overhead and thus can be more useful for domains where memory is a bottleneck.

5. Conclusions and Future Work

We presented a heuristic search framework (MHA*) that uses multiple inadmissible heuristics to simultaneously explore the search space, while preserving guarantees of completeness and sub-optimality bounds using a consistent heuristic. We described two variants of the MHA* approach, IMHA* where individual searches run independently and SMHA*, where the searches share the current path obtained to a state. Experimental results obtained on various domains demonstrate the efficacy of the proposed framework, especially for search spaces with large depression regions. Overall, the approach is simple (i.e., easy to implement) and yet very powerful, and we hope this will make it useful for solving complex planning problems in robotics and other domains.

In addition, while the initial results with MHA* are encouraging, we believe that there are plenty of directions for major improvements/extensions. One of the direction can be to use meta-reasoning to dynamically select which heuristic to explore, the current version uses round robin scheduling, however, more intelligent approaches can intelligently schedule the searches depending on their progress on a particular instance. If done efficiently such intelligent scheduling may also enable us to use a larger number of heuristics, which in turn can increase the coverage of the algorithms when solving really hard problems. Other possible future extensions on the algorithmic side include anytime version of MHA*, dynamic re-computation of heuristics, and parallel MHA*, all of which should enhance the capabilities of MHA* significantly. On the application side, we would like to investigate MHA*'s applicability for other planning domains, such as symbolic planning.

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