When seeing isn't believing: Influences of prior conceptions and misconceptions

Eliane Stampfer (stampfer@cs.cmu.edu)
Kenneth R. Koedinger (krk@cs.cmu.edu)
Human Computer Interaction Institute, 5000 Forbes Ave
Pittsburgh, PA 15213 USA

Abstract

Instruction often employs visual representations to support deep understanding. However, students' prior misconceptions may override the meaning in these scaffolds. We investigate fraction bars, a common representation intended to promote sense-making. Our prior work found that students often did not use the fraction bars effectively. This difficulty factors assessment compares four scaffold types: pictures only, two forms of pictures with numbers, and numbers only, to assess which interpretation steps were difficult. On equivalence items, students performed equally well with all scaffolds that included pictures, but worse with the numbers-only scaffold, indicating that fraction bars improved scores for equivalence. However, including numbers with the pictures decreased performance for fraction addition. Although students demonstrated competence with fraction bars in fraction equivalence, they did not transfer this knowledge to addition. These results suggest caution in designing and teaching representations for sense-making.

Keywords: graphical representation; fraction addition; symbolic fractions.

Many researchers strive to identify ways to support deep understanding, as it is thought to promote robust and adaptable learning. One strategy has been to use multiple representations, particularly ones that connect to students' prior knowledge and aid sense-making. However, there is little data on what representations will make sense to the students. Singapore textbooks and the NCTM standards, for example, advocate using concrete visual representations in mathematics as a bridge to more formal, abstract thinking (NCTM, 2013; Leinwand & Ginsburg 2007). But, perhaps we should question the benefits of these representations (Rittle-Johnson & Koedginer, 2001; Booth & Koedinger, 2012): Are they actually easy entry points for students?

Our tutors for 5th graders aim to support sense-making by providing conceptual representations as feedback, a strategy that appears effective with adults (Mathan & Koedinger, 2005; Nathan, 1998). In our fraction-addition tutor interfaces, equally-divided rectangles, or *fraction bars*, provide immediate feedback by dynamically showing the fractions that students enter numerically. We hypothesized that fraction bars would be a more intuitive representation than symbolic fractions, and having students input symbolic fractions and get feedback from fraction bars would prompt thinking on how the two representations were related. Also, we thought it would show students that the common mistake of adding both numerators and denominators was incorrect. We termed this feedback *grounded feedback* because it was grounded in student's prior knowledge, and grounded an unfamiliar representation (fraction symbols) in a more intuitive one (fraction bars). An initial think aloud study showed promise. The 5th grade participants seemed to understand what the fraction bars meant, and used them to find and correct fraction-addition errors (Stampfer, Long, Aleven, & Koedinger, 2011). An experimental study found learning benefits with a fraction bar tutor (interface in Figure 1) (Stampfer & Koedinger 2012). This tutor does not indicate explicitly if an individual step is right, but students cannot advance to the next problem until all steps have been solved correctly.

Although students learned from the tutor (improved from pre-test to post-test), process measures show incorrect interpretations of the fraction bars. Students often indicated they were done solving the problems even though the fraction bars did not line up. They clicked the "done" button on the tutor screen an average of about 2.5 times per problem (rather than the one necessary click). This finding revealed that one of our key assumptions about this form of grounded feedback for these students was not fully satisfied. It appeared that the fraction bar representations of addition were not as meaningful to all students as the think-aloud results suggested. Thus, we were led to investigate more deeply the cognitive

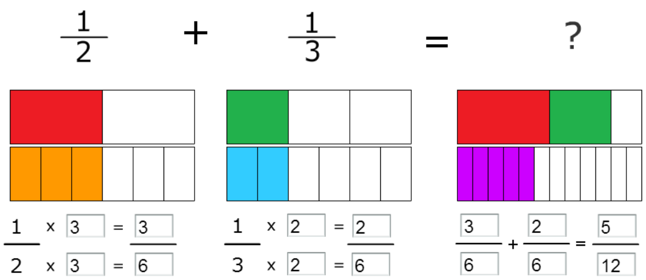

Figure 1: Fraction Addition Tutor. Top row of fractions and fraction bars are given, second row reflects students' inputs, typed in the boxes at the bottom. Text hints appear below when requested.

mechanisms required for processing these representations and, in particular, to attempt to identify the sticking points where student processing deviates from expectation. This difficulty factors assessment (cf., Koedinger, Alibali, & Nathan, 2008) examines how students understand fraction bars in the context of the fractions they represent; if this process changes depending on the topic (addition vs. equivalence); and how each processing step affects performance.

Difficulty Factors: Pictures and Numbers

Using a theoretical cognitive task analysis, we identified three likely skills needed to understand the fraction bar representations for fraction addition: 1) equal areas represent equal amounts; 2) the rectangular bars represent the symbolic fractions written above or below them; 3) if two shaded areas are equal, the fractions they represent are equal. We developed matched test items intended to isolate those skills (Figures 2-5). Fraction addition items presented a fully solved problem and students indicated whether it was solved correctly (true or false). Fraction equivalence items presented two fractions and students indicated if the first fraction was bigger than, equivalent to, or smaller than the second fraction. The four question presentations are intended to isolate the skills needed to make sense of the tutor interface in Figure 1. The pictures format (Figure 2) assesses if students know that the shaded rectangles use area to represent quantity, such that two rectangles with equal-sized

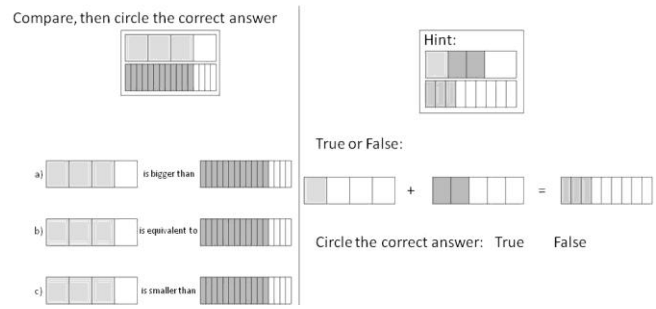

Figure 2: Pictures. Does area equal quantity?

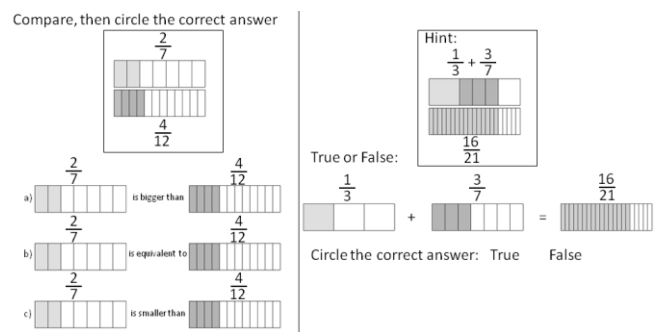

Figure 3: Pictures and Numbers. Are images comprehensible as fractions?

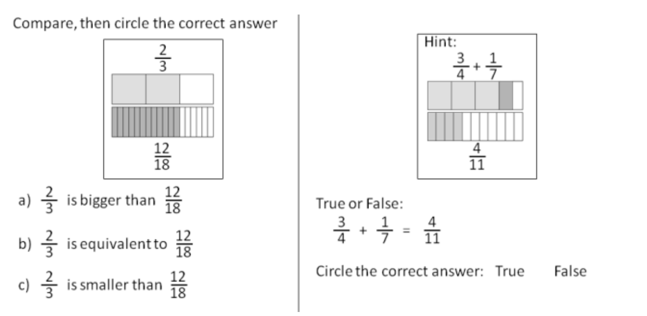

Figure 4: Half Pictures and Numbers. Can students map relationships from images to symbols?

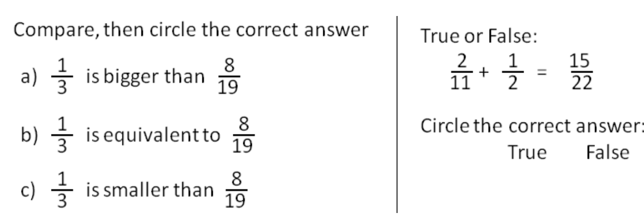

Figure 5: Numbers-Only Control. Can students evaluate solved problems?

shaded areas represent equal quantities. Pictures-and-numbers items (Figure 3) include fraction symbols with the fraction bars, to test if students can understand the fraction bars as representations of fractions. Half-pictures-and-numbers items (Figure 4) also include both fraction bars and fraction symbols, but only presents the fraction bars as the hint at the top of the problem. This determines if students can find the relationship between the two fraction bars, map that relationship to the symbolic fractions represented, and then select the relationship that the symbolic fractions have to each other. Numbers-only (Figure 5) provides a baseline for how well students can evaluate the equivalence and addition problems without fraction bars. Another pair of questions gives a baseline for translating a single fraction bar to a fraction symbol (e.g., when shown a rectangle divided in 5 parts with 3 of them shaded, the student should write 3/5).

Methods and Participants

155 fifth grade students from a local public school participated in this study during their normal school day. They were given 20 minutes for a 30-item assessment. The school tracked these classes, with 57 students in the highest track, 61 in the middle track and 37 in the lowest track.

Each test included 8 equivalence items and 8 addition items (one correctly solved and one incorrectly solved for each scaffold type). All addends in these items had unlike denominators. The sums in the incorrect addition items followed the popular misconception of adding both numerators and both denominators. Tests also included two single fraction bar items, one with numbers for how many pieces were shaded and how many total. Item presentations were counterbalanced with the specific numbers in the problems to avoid confounding. Item order was determined randomly and half of the tests were given with the order reversed. Questions were scored as 1 if correct and 0 otherwise.

Results: Scaffold Type Affects Performance

Scores on the single-fraction-bar items were near perfect (94% correct). Figure 6 shows the mean scores for the equivalence and addition items by

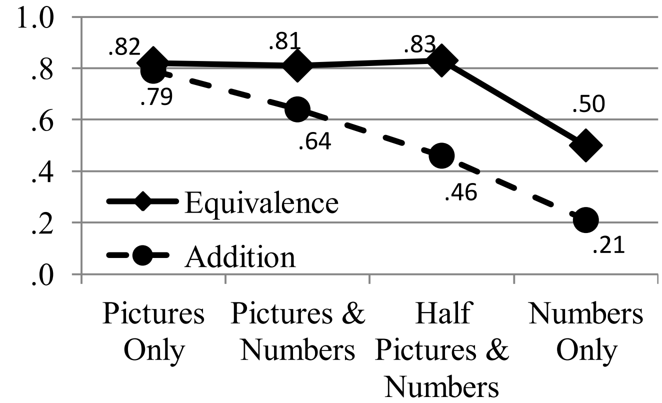

Figure 6: Mean scores (max. 1) on equivalence and addition items

scaffold type. Mean scores on the fraction equivalence items were high, with 81-83% correct for all scaffold types with pictures, and 50% for the numbers-only presentation. Equivalence items offered three options (bigger, equivalent, or smaller) so even the numbers-only score is well above 1/3 chance. Mean scores on the fraction addition items were lower (21% to 79%). These scores steadily decreased as the saliency of the numbers increased. Lower-than-chance results indicate that instead of guessing randomly on the more difficult scaffolds, students answered based on a systematic misconception. Blank answers were scored as 0 and they could reduce performance below the 50% chance rate. However, students were no more likely to skip the numbers-only addition items than the other addition items that included numbers (numbers-only addition was skipped 13 times, while half-pictures-and-numbers and pictures-and-numbers were skipped 14 times each).

There is a strong interaction effect between question type and scaffold type. We ran an ANOVA on the item scores: 3 (class tracking level: high, middle, low) x 4 (scaffold type: pictures, pictures and numbers, half pictures and numbers, numbers only) x 2 (item: equivalence or addition) with repeated measures for the scaffold type and item. With the Huynh-Feldt correction (since sphericity could not be assumed), results showed significant within-subjects effects for scaffold type and item, and a significant scaffold by item interaction (all $p<.0005$). Results also showed significant between-subjects effects for class tracking level, with parameter estimates indicating that higher-tracked students got higher scores.

The patterns in figure 6 suggest that all scaffold types with pictures have a similar effect for equivalence, but each scaffold type has a different effect for addition. To verify these hypotheses statistically, we ran separate ANOVAs on each tracking level for equivalence and addition scores, with scaffold type as a fixed factor and student as a random factor. For each of those analyses on the equivalence scores, scaffold was significant ($p<.0005$) and post-hoc Tukey tests showed that the numbers-only scaffold was significantly different from the other three ($p<.0005$). For each of those analysis on the addition scores, scaffold was again significant ($p<.0005$). Tukey tests for the middle track show significant differences among all scaffold types ($p<.01$). The lowest track did not have significant differences between half-pictures-and-numbers and numbers-only, likely a floor effect. The highest track did not have significant differences between pictures and pictures-and-numbers, likely a ceiling effect.

Figure 6 also suggests that addition with the pictures-only scaffold is no more difficult than equivalence with the pictures-only scaffold. To test this, we ran an ANOVA on the item scores for the pictures-only scaffold: 3 (class tracking level: high, middle, low) x 2 (item: equivalence or addition) with repeated measures for item. Results showed no significant difference for scores on the two question types ($p = .2$ with the Huynh-Feldt correction). Subsequent ANOVAs on each of the other scaffold types showed significant differences for scores on the two question types (all $p<.0005$ with the Huynh-Feldt correction).

Finally, we examined the effect of spatial reasoning on scores. One may hypothesize that when pictures are present, students would be more accurate when there is a large disparity in the area of the quantities being compared. To test this hypothesis, we calculated a disparity measure for each question where the two fractions were not equivalent or the two addends did not equal the sum. For the equivalence items, the disparity is the absolute value of the first fraction minus the second fraction. For the addition items, the disparity is the true sum of the addends minus the sum in the question. We ran separate ANOVAs for each question type, with scaffold type and disparity as fixed factors and student ID as a random factor. For both addition and equivalence, between-subject main effects were significant for scaffold type and student ID ($p<.0005$) but not for disparity ($p=.141$ for addition, $p=.888$ for equivalence), and there was no scaffold*disparity interaction ($p=.257$ for addition, $p=.136$ for equivalence). This indicates that disparity did not affect scores, and the effect of disparity did not change with scaffold type. Additionally, the equivalence questions all had smaller disparities than the addition questions (means: .06 for equivalence, .39 for addition), yet the equivalence questions were as easy or easier, further evidence that disparity did not affect scores.

Discussion: Fraction Bar Skills are Context-Based

Students were at ceiling for writing the symbolic fraction represented by a single fraction bar. Students were quite good at comparing two fractions and indicating if the first was greater than, equivalent to, or smaller than the second. Further, scores on these equivalence items were equally high for all scaffold types that included pictures.

On the equivalence items, students demonstrate competence with the three skills identified in the cognitive task analysis: equal areas represent equal quantities (pictures), the bars represent fractions (pictures and numbers), the relationship between the bars maps to the relationship between the fractions they represent (half pictures and numbers). Students were likely not solving these equivalence problems with the numbers alone, since numbers-only performance is much lower.

Surprisingly, these skills are not consistently demonstrated with fraction addition. Pictures-only scores are just as high with addition as they are with equivalence, indicating that the knowledge that equal areas represent equal quantities does transfer to addition. However, performance decreases steadily across pictures-and-numbers and half-pictures-and-numbers, suggesting difficulty both with understanding the bars in the context of fractions and mapping the relationship between the fraction bars to the relationship between the fraction symbols. Yet, the bars still increase performance above the numbers-only control (which has worse-than-chance scores).

We hypothesize that the temptation of the incorrect add-both-numerators-and-denominators strategy overrides the area-as-quantity reasoning that students demonstrate when the numbers are not shown. A cognitive-load hypothesis may predict that fraction symbols are distracting because they visually clutter the problem. In that case, scores with half pictures and numbers should be higher than pictures and numbers, since there is less information and less visual clutter. Yet, scores decrease, indicating that performance is not correlated with cognitive load.

Byrnes and Wasik (1991) discuss a theory that conceptual knowledge will prevent students from making certain procedural errors. In this theory, a "self-critic" (our name), evaluates procedural outcomes for conceptual errors. For example, if a student adds 3/4 and 1/7 and gets 4/11, their "self-critic" may reason that 4/11 cannot be right because it is less than half while 3/4 is greater. With the picture scaffolds, these steps are easier – instead of numeric mental operations, students can compare the fraction bars. Scores on the equivalence and the pictures-only addition items demonstrate students' skill in comparing fraction bars, yet they still seem to not use their "critic" on the fraction addition items with numbers.

Interestingly, Byrnes and Wasik argue against the self-critic theory, claiming that conceptual and procedural knowledge are not activated simultaneously in problem solving. Further, conceptual knowledge may precede procedural skill, so in some stages of learning conceptual knowledge would not be correlated with procedural performance. Instead, procedural skills improve through proper discrimination and generalization. To test these theories, they compared three instructional techniques for LCD fraction addition. One was procedural, and stressed that "you can't add fractions the way you add ordinary numbers." The other techniques added conceptually-based instruction (one with paper fraction bars) to that procedural instruction. Results showed that the conceptual methods did not improve learning above the purely procedural one. These findings suggest that aiding discrimination will improve procedural skill, and that skill is not enhanced further with brief conceptual instruction. These findings and the results from the fraction equivalence items suggest that students will not benefit from more conceptual instruction on fraction bars, even though they performed poorly on fraction addition items with fraction bars. Instead, they may benefit from support for separating whole-number and fraction addition. Alternatively, students may benefit from practice and support in invoking their "self-critic." However, these critics may be stifled by a misconception unrelated to fractions: the meaning of the equals sign.

McNeil et al. (2006) found that 6^{th}-8^{th} grade students looking at a problem such as $3 + 4 = 7$ were more likely to interpret the equals sign to mean 'write answer here' than 'both sides are equivalent.' Perhaps this misinterpretation of the equals sign in equations with operations interfered with students' internal "critic" in the addition items. Even when the pictures show the sum to be smaller than one of the addends, the student may not realize that the two sides of the equal sign are supposed to be equivalent. A "critic" that interprets "=" as 'write output of procedure here' may simply verify that the add-both-numerators-and-denominators strategy was executed well. In other words, the presence of numbers may not only prompt over-generalization of whole-number addition, but also interfere with students' interpretation of the equals sign and thus throw off the "critic."

Conclusion

These data imply that the usefulness of the fraction bar scaffold is dependent on the topic for which it is employed, and the specific combination of images and numbers. When naming fractions represented by individual fraction bars and solving equivalence problems with fraction bars, students were equally proficient whether the numeric symbols were present or not. However, for fraction addition, the fraction symbols were detrimental. The pictures-only addition problems may invite reasoning based on conceptual understanding (the sum of two areas cannot be smaller than either addend), while the presence of fraction symbols may invite procedural problem solving that is initially divorced from the underlying concepts.

This DFA study suggests that students' difficulty with dynamic fraction bars in a tutoring system was due to the specific addition context. More broadly, it suggests caution in the design and use of conceptual scaffolds for math problems. Students may demonstrate proficiency with a scaffold in one domain without being able to transfer those skills, even to a closely related domain. Procedural misconceptions may override the conceptual reasoning these scaffolds attempt to induce. Perhaps students need instruction to support their "self-critics" in checking procedural outcomes against conceptual knowledge. Or, perhaps students require certain domain-specific knowledge before their "self-critics" are triggered. Our future work with fraction bars will explore the effect of metacognitive "self-critic" training and domain-specific instruction on the meaning of the equals sign.

Acknowledgments

We thank our participants, their teachers, Jo Bodner, and Stephanie Butler. This work was supported in part by the Pittsburgh Science of Learning Center through NSF award SBE-0836012, and by a Graduate Training Grant awarded to Carnegie Mellon University by the Department of Education (# R305B090023).

References

Booth, J. L., & Koedinger, K. R. (2012). Are diagrams always helpful tools? Developmental and individual differences in the effect of presentation format on student problem solving. *British Journal of Educational Psychology 82(3)* 492-511.

Byrnes, J.P., & Wasik, B.A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology 27(5)* 777–786.

Koedinger, K.R., Alibali, M.W., & Nathan, M.J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science 32(2)* 366–97.

Leinwand, S., & Ginsburg, A.L. (2007). Learning from Singapore Math. *Making Math Count 65* (3) 32-36.

Mathan, S., & Koedinger, K. R. (2005). Fostering the Intelligent Novice: Learning From Errors With Metacognitive Tutoring. *Educational Psychologist*, *40(4),* 257–265.

McNeil, N.M., Grandau, L., Knuth, E.J., Alibali, M.W., Stephens, A.C., Hattikudur, S., & Krill, D.E. (2006). Middle-School Students' Understanding of the Equal Sign: The Books They Read Can't Help. *Cognition and Instruction 24(3)* 367–385.

Nathan, M. J. (1998). Knowledge and Situational Feedback in a Learning Environment for Algebra Story Problem Solving. *Interactive Learning Environments 5*, 135–159.

National Council of Teachers of Mathematics (2001). Navigating Through Algebra in Grades 6-8, as cited in http://illuminations.nctm.org/LessonDetail.aspx?ID=L605

Rittle-Johnson, B., & Koedinger, K. (2001). Using cognitive models to guide instructional design: The case of fraction division. In J. Moore & K. Stenning (Eds.), *Proceedings of the 23rd Annual Conference of the Cognitive Science Society*. (857-862). Mahwah, NJ, Erlbaum.

Stampfer, E., & Koedinger, K.R. (2012) Tradeoffs between immediate and future learning. Presented at the European Association for Research on Learning and Instruction Special Interest Groups 6&7 Conference. Bari, Italy.

Stampfer, E., Long, Y., Aleven, V., & Koedinger, K.R. (2011) Eliciting intelligent novice behaviors with grounded feedback in a fraction addition tutor. In G. Biswas, S. Bull, J. Kay, & A. Mitrovic, (Eds.), *Proceedings of the 15th International Conference of Artificial Intelligence in Education* (560-562) Auckland, New Zealand: International AIED Society.