Some Sequential Algorithms are Almost Always Parallel

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?

are Almost Always

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Parallel

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Parallel

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Joint work with Jeremy Fineman, Phil Gibbons, Yan Gu, Julian Shun, and Yihan Sun [BFS SPAA'12], [BFGS PPOPP'12], [SGuBFG SODA'15], [BGuSSu SPAA'16].

Iterative Sequential Algorithms

```
for i = 1 to n
  do something
```

Iterative Sequential Algorithms

```
for i = 1 to n
  a[i] = f(a[i-1])
```

Iterative Sequential Algorithms

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for i = 1 to n
    a[i] = f(a[i-1])
```

Fully Sequential (for arbitrary f)
Each iteration depends on all previous iterations.

Iterative Sequential Algorithms

```
for i = 1 to n
    a[i] = a[i] + 1
```

Iterative Sequential Algorithms

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```

Fully "parallel"

No dependences among iterations.

Iterative Sequential Algorithms

```
for i = $\sqrt{n}+1$ to $n$
    a[i] = f(a[i-$\sqrt{n}$])
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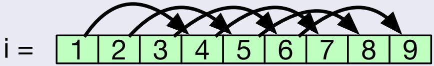

$i \rightarrow j$ means j depends on i

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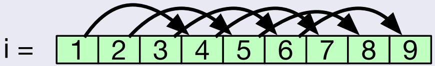

$i \rightarrow j$ means j depends on i

Partially parallel.

Some dependences, but they are not affected by the data.
In this case dependence depth is $\sqrt{n}$.

Iterative Sequential Algorithms

?

```
$A =$ an input array of length $n$
for $i = n - 1$ downto 0
  H[$i$] = rand($\{0, \ldots, i\}$)
  swap(A[H[$i$]],A[$i$])
```

Iterative Sequential Algorithms

Knuth's shuffle to generate a random permutation of A

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i =	0	1	2	3	4	5	6	7
A[i]	a	b	c	d	e	f	g	h
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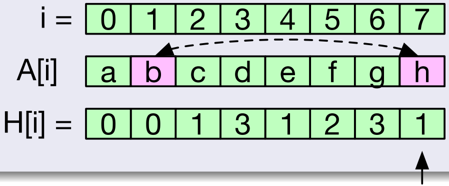

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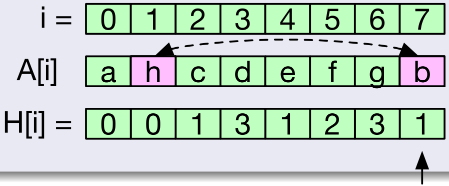

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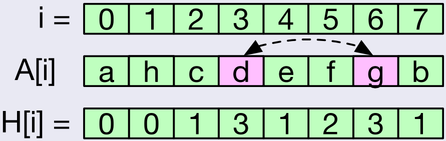

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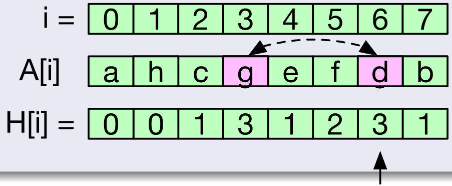

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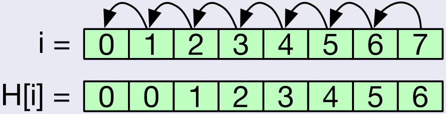

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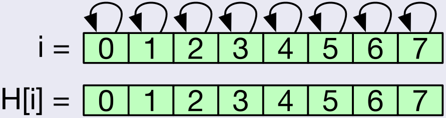

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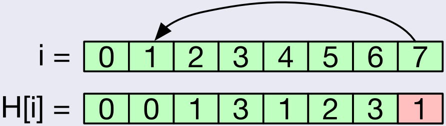

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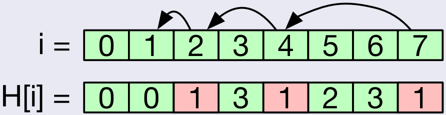

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Dependences **depend on data**.

Question: What can we say about dependence depth over the **random choices**?

Iterative Sequential Algorithms

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Dependences **depend on data**.

Question: What can we say about dependence depth over the **random choices**?

Answer [SGuBFG'15]: $O(\log n)$ w.h.p.

Iterative Sequential Algorithms

?

```
undirected graph $G = (V, E)$, $S[1, \ldots, n]$ = unknown
for $i = 1$ to $|V|$
    if for any earlier neighbor $v_j$ of $v_i$, $S[j]$ = in
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Iterative Sequential Algorithms

Greedy Maximal Independent Set

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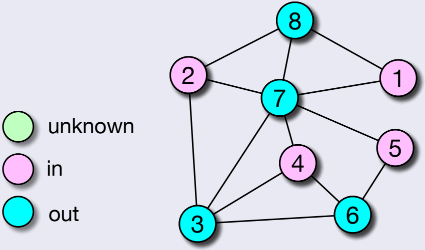

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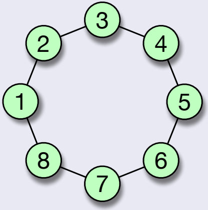
Dependences for this simple cycle graph?



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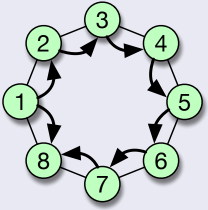
Fully sequential



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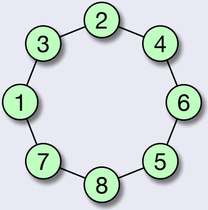
Same graph, different ordering of V. Dependences now?



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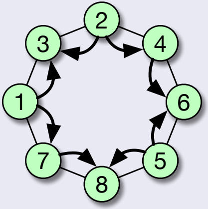

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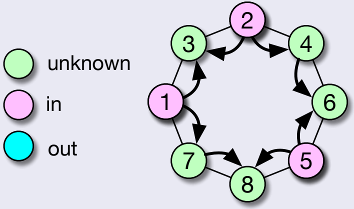

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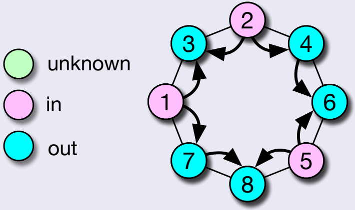
Fully parallel order (two rounds)



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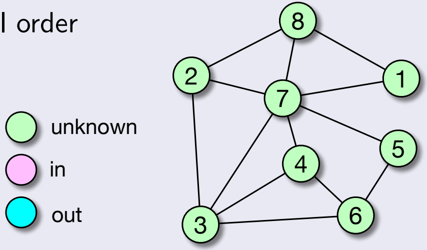
Partially parallel order



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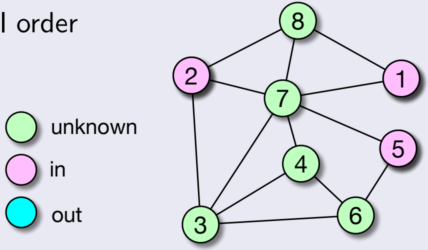

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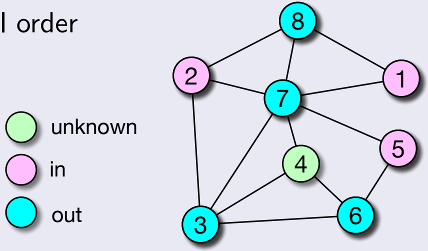

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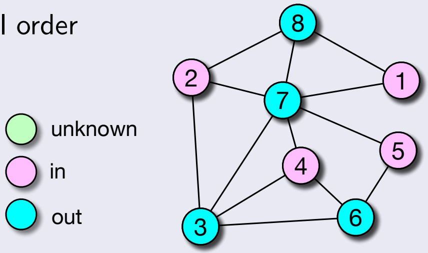

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Question: what is the dependence depth for a **random order of the vertices**?

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Answer [BFS'12]: $O(\log^2 n)$
(w.h.p. for random ordering of V, i.e. if we randomly permute V)

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Answer [BFS'12]: $O(\log^2 n)$
(w.h.p. for random ordering of V, i.e. if we randomly permute V)
Open problem: $O(\log n)$ w.h.p.?

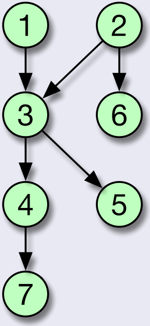
Dependence Depth

```
for i = 1 to n
    do something
```

Simple model:

- Each iterate is a vertex
- $i \to j$ means j depends on i

Iterations



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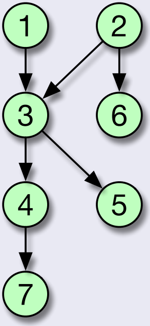
Iteration Depth:
the longest chain of dependences.

Iterations



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Iteration Depth:
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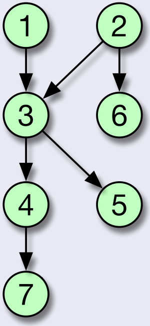
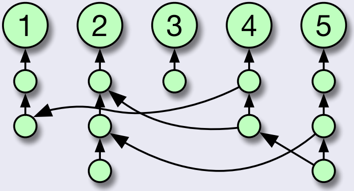
Overall Depth:
weighted by depth of each iteration.

Iterations



Dependence Depth

Nested iterations

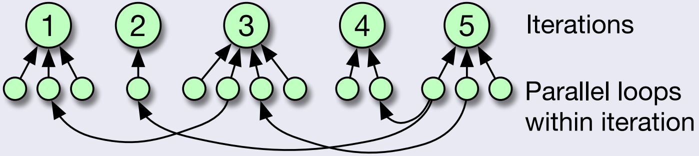


Iterations

Sequential loops
within iterations

Dependence Depth

Nested iterations



Only Two Examples?

1. Knuth shuffle
2. Greedy MIS

More Iterative Sequential Algorithms

Greedy Maximal Matching

```
Graph $G = (V, E)$ and
$A[1, \ldots, |V|]$ = true        // $A[i]$ if vertex $i$ is available
$M[1, \ldots, |E|]$ = false       // $M[j]$ if edge $j$ is in matching
for $i = 1$ to $|E|$
    $(u, v) = E[i]$
    if ($A[u]$ and $A[v]$)
    then $M[i]$ = true, $A[u]$ = false, $A[v]$ = false
```

Iteration Depth [BFS'12]: $O(\log^2 |V|)$
(w.h.p. for random ordering of E)

More Iterative Sequential Algorithms

List contraction

```
$A$ = an array containing $n$ links of doubly linked lists
for $i = 0$ to $n - 1$
    $A[i]$.next.previous = $A[i]$.previous
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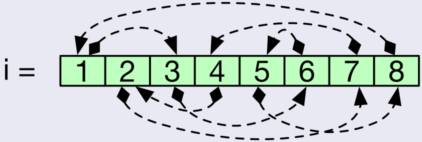

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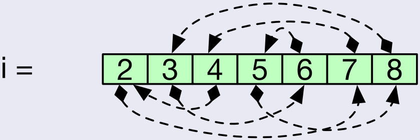

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for $i = 0$ to $n - 1$
    $A[i]$.next.previous = $A[i]$.previous
    $A[i]$.previous.next = $A[i]$.next
```

Applications of:

- length of lists or list ranking
- size or number of cycles in a permutation
- Euler tour, biconnectivity,..

More Iterative Sequential Algorithms

List contraction

```
$A$ = an array containing $n$ links of doubly linked lists
for $i = 0$ to $n - 1$
    $A[i]$.next.previous = $A[i]$.previous
    $A[i]$.previous.next = $A[i]$.next
```

Iteration Depth [SGuBFG'15]: $O(\log n)$
(w.h.p. for random ordering of A)

More Iterative Sequential Algorithms

Sorting by insertion into a binary search tree (BST)

```
$A =$ an array of keys
$T$ = empty binary tree
for $i = 1$ to $n$
  BST_insert($T, A[i]$)
```

More Iterative Sequential Algorithms

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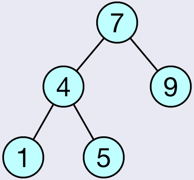
Example:



insert(2), insert(8) – no dependence

insert(2), insert(3) – dependence

More Iterative Sequential Algorithms

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Overall Depth: $O(\log n)$

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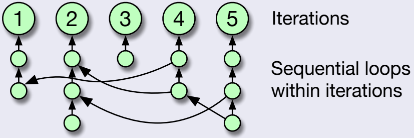
Overall Depth: $O(\log n)$

Analysis requires sub-iteration dependences, otherwise $O(\log^2 n)$.



More Iterative Sequential Algorithms

two-dimensional linear programming

```
Constraints (lines) $C = c_1, \ldots, c_n$
$p = (\infty, 0)$
for $i = 1$ to $n$
  if $p$ violates $c_i$
    $p$ = min intersection of $c_1, \ldots, c_{i-1}$ along $c_i$
```

More Iterative Sequential Algorithms

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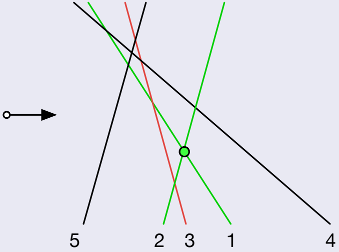

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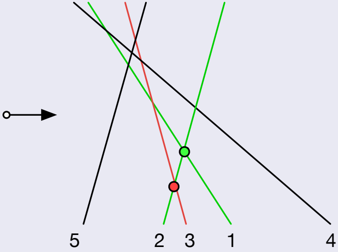

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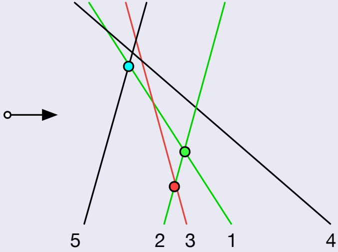

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Total work: $O(n)$ (w.h.p. for random ordering of C)

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Iteration Depth [BGuSSu'16]: $O(\log n)$
(w.h.p. for random ordering of C)

Overall Depth: $O(\log(n) \log\log(n))$
($\log\log(n)$ term for min intersection)

More Iterative Sequential Algorithms

Delaunay triangulation

```
points $P = p_1, \ldots, p_n$
$T$ = {boundingTriangle of $P$}
for $i = 1$ to $n$
  for $t$ in conflictSet($p_i$, $T$)
    replace $t$ with new triangle(s) in $T$
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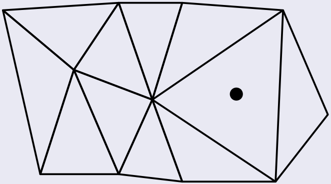

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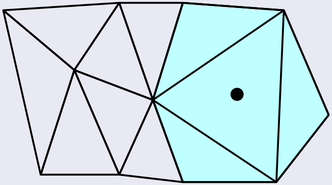

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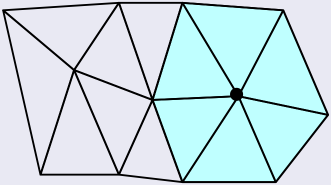

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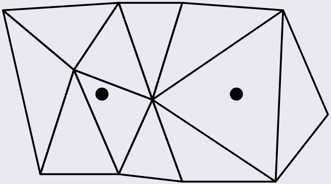

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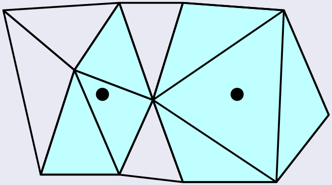

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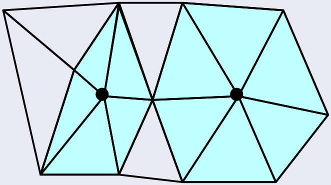

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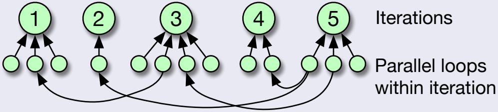
Analysis requires allowing subiterations to proceed independently.



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Iteration Depth [BGuSSu'16]: $O(\log n)$
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Parallel incremental Delaunay is **widely used in practice**, but it was not previously known whether it is theoretically efficient.

More Iterative Sequential Algorithms

Graph Connectivity using union-find

```
graph  $G = (V, E)$
F  =  a union find data structure on  $V$
for  $i = 1$  to  $|E|$
    $u$ = F.find($E[i].u$)
    $v$ = F.find($E[i].v$)
    if  ($u \neq v$)  then  F.union($u, v$)
```

More Iterative Sequential Algorithms

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```

Iteration Depth: TDB (open problem)

Some Sequential Algorithms

are Almost Always

Parallel

~~Some~~ Many Sequential Algorithms

- Iterative sequential algorithms such as:
  Knuth shuffle, greedy MIS, greedy maximal matching, list contraction, tree contraction, linear programming, Delaunay triangulation

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are Almost Always

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Parallel

- Polylogarithmic dependence depth

Why care?

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Intellectual curiosity

- Lots of work on parallel algorithms, but perhaps sequential ones are already parallel

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Application to parallel and distributed algorithms (Theory)

- Surprisingly simple solutions to basic problems
- Possible way to attack new problems
- Theoretical justification to current practice

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- Lots of work on parallel algorithms, but perhaps sequential ones are already parallel

Application to parallel and distributed algorithms (Theory)

- Surprisingly simple solutions to basic problems
- Possible way to attack new problems
- Theoretical justification to current practice

Application to parallel and distributed algorithms (Practice)

- Fast and simple code
- Generic techniques to parallelize code
- Determinacy

How to Analyze Dependence Depth

Knuth Shuffle: Iteration Depth

```
for $i = n - 1$ downto 0
    H[$i$] = rand($\{0, \ldots, i\}$)
    swap(A[H[$i$]],A[$i$])
```

i =	0	1	2	3	4	5	6	7
H[i] =	0	0	1	3	1	2	3	1

Knuth Shuffle: Iteration Depth

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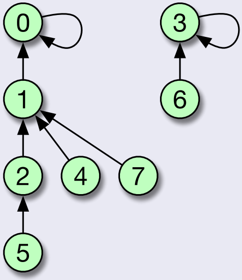

Chosen locations

Knuth Shuffle: Iteration Depth

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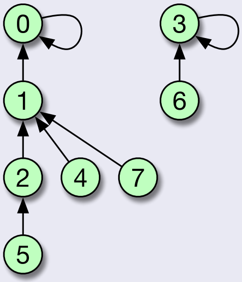

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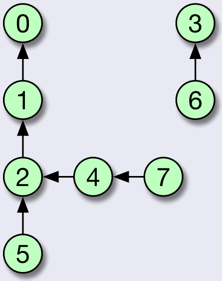

Actual dependences

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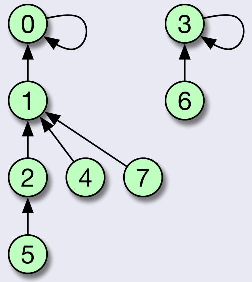

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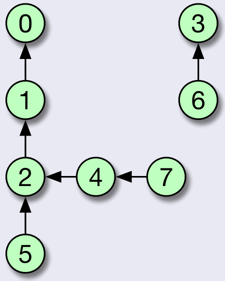

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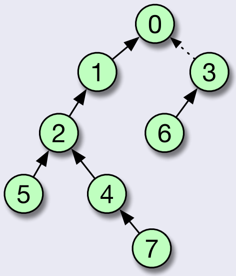
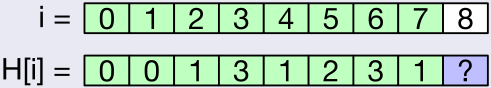


Rearranged

Knuth Shuffle: Iteration Depth

Adding one more, i.e., proof by induction



i =	0	1	2	3	4	5	6	7	8
H[i] =	0	0	1	3	1	2	3	1	?

Knuth Shuffle: Iteration Depth

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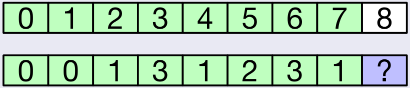

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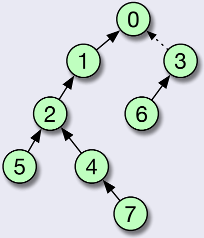

Not including 8

Knuth Shuffle: Iteration Depth

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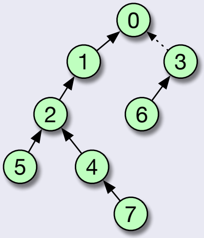

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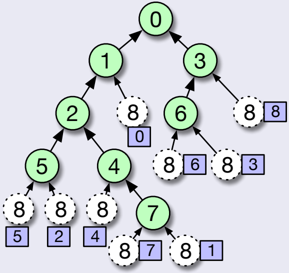

Possible positions of 8

Knuth Shuffle: Iteration Depth

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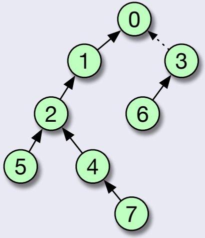

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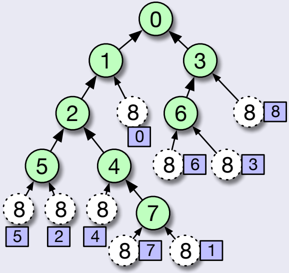

Possible positions of 8

All equally likely, so equivalent to random BST.

Maximal Independent Set, Bound on Depth

```
graph $G = (V, E)$, $S[1, \ldots, n]$ = unknown, $n = |V|$
for $i = 1$ to $n$
    if for any earlier neighbor $v_j$ of $v_i$, $S[j]$ = in
    then $S[i]$ = out else $S[i]$ = in
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Known results

- Lexicographically first MIS is P-complete [Cook '85]
- $O(\log^2 n)$ dependence depth for random graphs w.h.p. [Coppersmith, Raghavan, Tompa '89]
- $O(\log n)$ depth for random graphs w.h.p. [Calkin, Frieze '90]
- $O(\log^2 n)$ depth for arbitrary graph in random order
- Many parallel algorithms (e.g. Luby).

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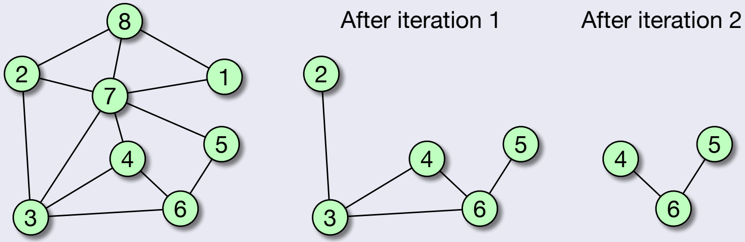
```

Definition (Residual graph on step *i*)

The graph that is left after step *i*



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Lemma (Degree)

After step i, the maximum degree in the residual graph is $O(n \log n / i)$ w.h.p. (over orderings of V).

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For example:

For $i = n/2$ (half done), the max degree is $O(\log n)$ w.h.p.

Maximal Independent Set, Bound on Depth

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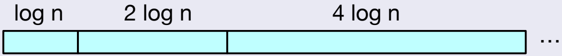
Proof outline.

Consider a vertex with degree larger than d (in residual graph) on step i. The probability of selecting one of the neighbors on each step j ($\leq i$) is at least d/n. The probability it survives all steps j is therefore at most $(1 - d/n)^i$, leading to the result. ☐

Maximal Independent Set, Bound on Depth

Consider increasing sized blocks of the iterations



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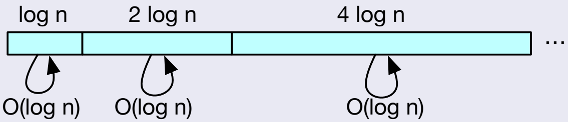

w.h.p. no path within a block is greater than $O(\log n)$

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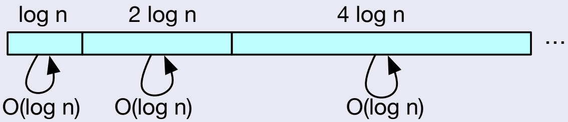

Proof outline: For each block *i*

- Prob. of edge beween two iterations is at most $\frac{1}{2^i}$ (by Degree Lemma)
- number of paths of length l is $\binom{2^i logn}{l}$

By the union bound the prob. of any path of length l is at most

$$\left(\frac{1}{2^i}\right)^l \binom{2^i logn}{l} < \left(\frac{1}{2^i}\right)^l \left(\frac{e2^i \log n}{l}\right)^l = \left(\frac{e \log n}{l}\right)^l$$

Therefore probability is very small that $l > 2e \log n$.

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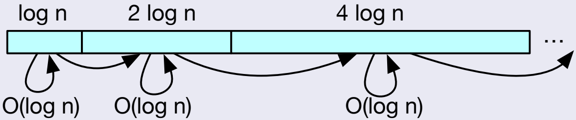

Summary

Since path within each block is $O(\log n)$, and number of blocks is $O(\log n)$, total depth is $O(\log^2 n)$.

Can be improved to $O(\log n \log d_{max})$ by picking blocks of size $2^i \frac{d_{max}}{n} \log n$.

Concrete Algorithms and Implementation

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Sufficient condition (true for all our examples):

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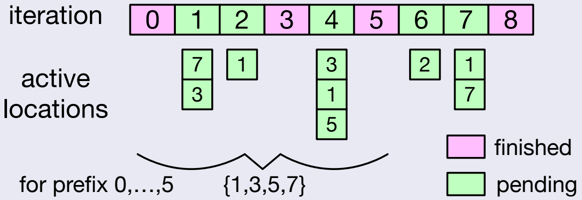
2. pending iterations in every prefix $0, \ldots, i$ only update $\cup_{j \in [i]} l_j$.



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Suggests an implementation strategy:

While there are pending iterations:

reserve: in parallel pending iterations find active locations and mark them

commit: in parallel each pending iteration runs, but aborts if it depends on an active previous location

Implication For Algorithms

Parallel (Priority PRAM)

	Work	Depth (Time)
MIS	$O(\|E\|)$	$O(\log^2 \|V\| \log \Delta)$
Maximal Matching	$O(\|E\|)$	$O(\log^2 \|V\| \log \Delta)$
BST Sort	$O(n \log n)$	$O(\log n)$
Knuth Shuffle	$O(n)$	$O(\log n \log^* n)$
List Contraction	$O(n)$	$O(\log n \log^* n)$
2d Linear Programming	$O(n)$	$O(\log n)$
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All are work efficient. Δ = maximum degree

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Distributed (CONGEST model)

	Rounds
MIS	$O(\log \|V\| \log \Delta)$
Maximal Matching	$O(\log \|V\| \log \Delta)$

Implementation: Speculative For

```
struct step {
  bool reserve(int i) {
    reserves locations that will be written by iteration $i$}
  bool commit(int i) {
    checks locations iteration $i$ depends on, and runs if safe}};
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speculative_for(Step(..), 0, n);  // $0, \ldots, n =$ range of iterations
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`speculative_for` will repeat the following until done:

- picks a prefix of remaining (pending) iterations
- in parallel runs the `reserve` on the prefix
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Can dynamically choose size of prefix.

Knuth Shuffle Code

```
struct knuth_step {
  bool reserve(int i) {
    write_min(R[i], i); write_min(R[H[i]], i);
    return 1; }
  bool commit (int i) {
    int h = H[i];
    if(R[H[i]] == i) {
      if(R[i] == i) {swap(A[i],A[H[i]]); R[i] = inf; return 1;}
      R[H[i]] = inf;}
    return 0; }
};
```

```
speculative_for(knuth_step(..), 0, n);
```

MIS Code

```
struct mis_step {
  bool reserve(int i) {
    flag = In;
    for (int j = 0; j < G[i].degree; j++) {
      int ngh = G[i].Neighbors[j];
      if (ngh < i) {
        if (S[ngh] == In) { flag = Out; return 1;}
        else if (S[ngh] == Unknown) flag = Unknown; }}
    return 1;}

  bool commit(int i) { return (S[i] = flag) != Unknown;}
};
```

```
speculative_for(mis_step(..), 0, n);
```

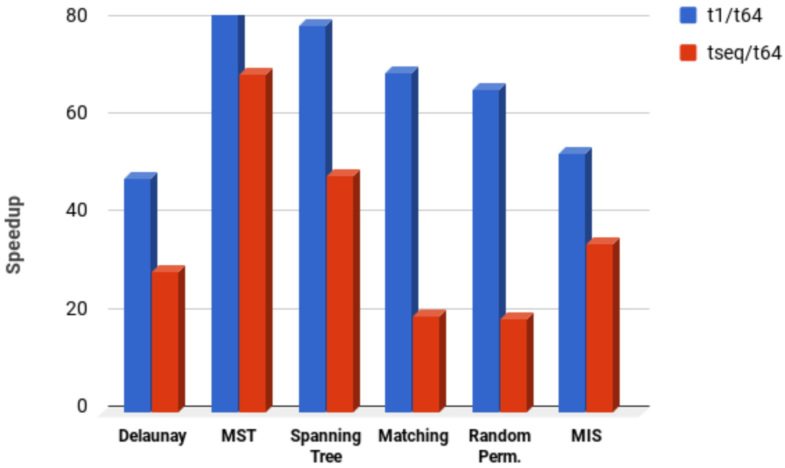
Spanning Tree Code

```
struct union_find_step {
  bool reserve(int i) {
    u = UF.find(E[i].u);
    v = UF.find(E[i].v);
    if (u > v) swap(u,v);
    if (u != v) { write_min(R[v], i); return 1;
    } else return 0; }

  bool commit(int i) {
    if (R[v] == i) { UF.link(v, u); return 1; }
    else return 0; }
};
```

```
speculative_for(union_find_step(..), 0, m);
```

Timings on a 64-core Xeon Phi



tseq = best sequential algorithm
t1 = time on one core
t64 = time on all cores

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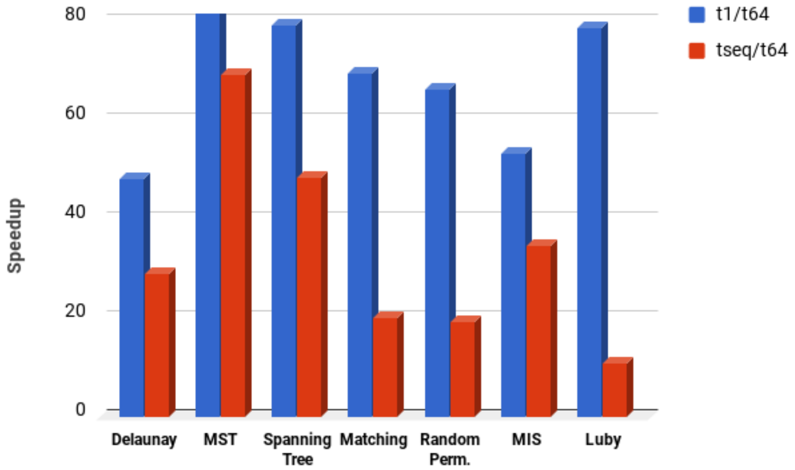

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End Matter

Conclusions

1. Many sequential algorithms are “inherently” parallel, at least when randomly ordering.
   - Perhaps should be thinking of algorithms more abstractly in terms of their dependence graph instead of specific model.
2. Can often take advantage of the parallelism using reservations
3. Resulting code is **simple**, **fast**, and **deterministic**

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Open Questions

1. Depth of MIS
2. Resolving dependences in a more general context.