Graduate AI

Lecture 24:
Social Choice III

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CONDORCET STRIKES AGAIN

• For Condorcet [1785], the purpose of voting is not merely to balance subjective opinions; it is a collective quest for the truth
• Enlightened voters try to judge which alternative best serves society
• For $m = 2$ the majority opinion will very likely be correct
• Realistic in trials by jury, but also in the pooling of expert opinions, or in human computation!
Welcome to eterna!
You play by designing RNAs, tiny molecules at the heart of every cell.
AI-Driven Decisions

RoboVote is a free service that helps users combine their preferences or opinions into optimal decisions. To do so, RoboVote employs state-of-the-art voting methods developed in artificial intelligence research. Learn More.

Poll Types

RoboVote offers two types of polls, which are tailored to different scenarios; it is up to users to indicate to RoboVote which scenario best fits the problem at hand.

Objective Opinions

In this scenario, some alternatives are objectively better than others, and the opinion of a participant reflects an attempt to estimate the correct order. RoboVote’s proposed outcome is guaranteed to be as close as possible — based on the available information — to the best outcome. Examples include deciding which product prototype to develop, or which company to invest in, based on a metric such as projected revenue or market share. Try the demo.

Subjective Preferences

In this scenario participants’ preferences reflect their subjective taste; RoboVote proposes an outcome that mathematically makes participants as happy as possible overall. Common examples include deciding which restaurant or movie to go to as a group, which destination to choose for a family vacation, or whom to elect as class president. Try the demo.

Ready to get started?

CREATE A POLL
Condorcet’s noise model

- True ranking of the alternatives
- Voting pairwise on alternatives, each comparison is correct with prob. \( p > 1/2 \)
- Results are tallied in a voting matrix
- **Poll 1:** What is the Borda score of alternative \( b \)?
  1. 5
  2. 8
  3. 10
  4. 16

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Condorcet’s ‘solution’

- Condorcet’s goal: find “the most probable” ranking
- Condorcet suggested: take the majority opinion for each comparison; if a cycle forms, “successively delete the comparisons that have the least plurality”
- In example, we delete $c > a$ to get $a > b > c$

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**Condorcet’s ‘solution’**

- With four alternatives we get ambiguities
- In example, order of strength is $c > d$, $a > d$, $b > c$, $a > c$, $d > b$, $b > a$
- Delete $b > a \Rightarrow$ still cycle
- Delete $d > b \Rightarrow$ either $a$ or $b$ could be top-ranked

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Condorcet’s ‘solution’

- Did Condorcet mean we should reverse the weakest comparisons?
- Reverse $b > a$ and $d > b \Rightarrow$ we get $a > b > c > d$, with 89 votes
- $b > a > c > d$ has 90 votes (only reverse $d > b$)

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EXASPERATION?

• “The general rules for the case of any number of candidates as given by Condorcet are stated so briefly as to be hardly intelligible . . . and as no examples are given it is quite hopeless to find out what Condorcet meant” [Black 1958]

• “The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ... no amount of examples can convey an adequate impression of the evils” [Todhunter 1949]
Young’s solution

• $M =$ matrix of votes

• Suppose true ranking is $a > b > c$; prob of observations $\Pr[M \mid >]$:
  $\binom{13}{8} p^8 (1-p)^5 \cdot \binom{13}{6} p^6 (1-p)^7 \cdot \binom{13}{11} p^{11} (1-p)^2$

• For $a > c > b$, $\Pr[M \mid >]$ is
  $\binom{13}{8} p^8 (1-p)^5 \cdot \binom{13}{6} p^6 (1-p)^7 \cdot \binom{13}{2} p^2 (1-p)^{11}$

• Coefficients are identical, so $\Pr[M \mid >] \propto p^{\#agree} (1-p)^{\#disagree}$

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YOUNG’S SOLUTION

- \( \Pr[\succ | M] = \frac{\Pr[M\succ] \cdot \Pr[\succ]}{\Pr[M]} \)
- Assume uniform prior over \( \succ \), \( \Pr[\succ] = \frac{1}{m!} \)
- Must maximize \( \Pr[M | \succ] \)
- The optimal rule maximizes number of agreements with voters on pairs of alternatives
- This rule is called the Kemeny rule
THE KEMENY RULE

• The Kendall tau distance between $>$ and $>'$ is
  \[ d_{KT}(>,>') = |\{(a, b) \in A^2 | (a > b) \land (b >' a)\}| \]

• The Kemeny rule chooses the ranking that minimizes the sum of Kendall tau distances to the preference profile

• Theorem [Bartholdi, Tovey, Trick 1989]: Computing the Kemeny ranking is NP-hard
THE KEMENY RULE

• Typically formulated as an IP: for every \( a, b \in A \), \( x_{(a,b)} = 1 \) iff \( a \) is ranked above \( b \), and

\[
    w_{(a,b)} = |\{i \in N \mid a >_i b\}|
\]

Minimize \( \sum_{(a,b)} x_{(a,b)} w_{(b,a)} \)
Subject to
For all distinct \( a, b \in A \), \( x_{(a,b)} + x_{(b,a)} = 1 \)
For all distinct \( a, b, c \in A \), \( x_{(a,b)} + x_{(b,c)} + x_{(c,a)} \leq 2 \)
For all distinct \( a, b \in A \), \( x_{(a,b)} \in \{0,1\} \)
THE MALLOWS MODEL

• Same as Condorcet’s model, but votes are rankings
• Defined by parameter $\phi \in (0,1]$
• Probability of a voter casting the vote $\succ'$ given true ranking $\succ$ is

$$\Pr[\succ' \mid \succ] = \frac{\phi^{d_{KT}(\succ',\succ)}}{\sum_{\succ'',} \phi^{d_{KT}(\succ'',\succ)}}$$

• Kemeny still gives the MLE ranking
\[
\Pr[>'] = \frac{\phi^{d_{KT}}(>,>)}{\sum_{>,''} \phi^{d_{KT}}(>,>)}
\]

What is the relation between \( \phi \) in the Mallows model, and \( p \) in the Condorcet model?
The Mallows Model

- How can we sample a vote?
- Suppose the true ranking is \( a > b > c \)
- Repeated insertion model:

\[
p_{ij} = \phi^{i-j} \cdot \frac{1-\phi}{1-\phi^i} \quad \text{for } j \leq i \leq m,
\]

Theorem [Doignon et al. 2004]:

By setting \( p_{ij} = \phi^{i-j} \cdot \frac{1-\phi}{1-\phi^i} \) for \( j \leq i \leq m \), RIM induces the same distribution over rankings as Mallows.
IS MALLOWS REALISTIC?

[Drag these down to the gray area below.]

A: 5 7 2
   8 1 3
   4 6

B: 7 4 6
   1 2
   8 5 3

C: 7 5 1
   2 3 6
   8 4

D: 2 4 3
   7 5
   8 1 6

Closest to solution (Fewest moves)
Farthest from solution (Most moves)

Mean total mistakes

Minimum moves from goal state

Plurality
Borda
Kemeny
Maximin
Thurstone

[Mao et al. 2013]
Random utility models

- Parameters $\theta = (\theta_1, \ldots, \theta_m)$
  - $m =$ number of alternatives
  - Each alternative $x_j$ modeled by utility distribution $D(\theta_j)$
- A voter’s utility $U_j$ for alternative $x_j$ is drawn independently from $D(\theta_j)$
- Voters rank alternatives by $U_1, \ldots, U_m$:
  \[
  \Pr[x_2 > x_1 > x_3 \mid \theta_1, \theta_2, \theta_3] = \Pr_{U_j \sim D(\theta_j)}[U_2 > U_1 > U_3]
  \]
Random utility models

Generating a single vote

\[ x_2 > x_3 > x_1 \]
Random utility models

Voter 1
\[ x_3 > x_2 > x_1 \]

Voter 2
\[ x_3 > x_1 > x_2 \]

Voter 3
\[ x_2 > x_3 > x_1 \]

Generating a preference profile

\[ \Pr[>_1, \ldots, >_n \mid \theta] = \prod_{i \in N} \Pr[>_i \mid \theta] \]
The Thurstone Model

• Defined by a normal distribution
  ○ For each \( x_j, \theta_j = (\mu_j, \sigma_j) \)
  ○ \( D(\theta_j) = \mathcal{N}(\mu_j, \sigma_j^2) \)

• Computing \( \Pr[\succ | \theta] \) believed to be hard
THE PLACKETT-LUCE MODEL

- Defined by a Gumbel distribution
  - For each \( x_j, \theta_j = (\mu_j, \beta_j) \)
  - \( D(\theta_j) = G(\mu_j, \beta_j) \)
- Equivalently, there exist weights \( w_1, \ldots, w_m \)
such that \( \Pr[x_{j_1} > x_{j_2} \cdots > x_{j_m} | \mathbf{w}] \) is given by

\[
\frac{w_{j_1}}{w_{j_1} + \cdots + w_{j_m}} \cdot \frac{w_{j_2}}{w_{j_2} + \cdots + w_{j_m}} \cdots \frac{w_{j_m-1}}{w_{j_{m-1}} + w_{j_m}}
\]
The Plackett-Luce Model

Urn interpretation

$$\Pr[a > c > d \mid (4,3,2)] = \frac{4}{9} \cdot \frac{2}{5}$$
BEYOND SOCIAL CHOICE

• We previously interpreted pairwise comparisons as voters comparing alternatives
• But these comparisons can be the results of competitions between players
• In these situations, we typically want to update our estimates of player ratings online
• The famous **Elo system** originally used the Thurstone model
TrueSkill™ system used to rank Halo players
Also based on the Thurstone model
[Herbrich et al. 2006]
SUMMARY

• Terminology:
  o Models: Condorcet, Mallows, random insertion, Thurstone, Plackett-Luce
  o Kendall tau distance
  o The Kemeny rule

• Algorithms:
  o IP for Kemeny

• Big ideas:
  o Voting as search for truth