Graduate AI
Lecture 18:
Game Theory I

Teachers:
Zico Kolter
Ariel Procaccia (this time)
MULTIAGENT SYSTEMS
Multiagent systems

Chapters of the Shoham and Leyton-Brown book:

1. Distributed constraint satisfaction
2. Distributed optimization
3. Games in normal form
4. Computing solution concepts of normal-form games
5. Games with sequential actions
6. Beyond the normal and extensive forms
7. Learning and teaching
8. Communication
9. Social choice
10. Mechanism design
11. Auctions
12. Coalitional game theory
13. Logics of knowledge and belief
14. Probability, dynamics, and intention

Legend:
- “Game theory”
- Not “game theory”
MULTIAGENT SYSTEMS

Mike Wooldridge’s 2016 publications:

- [c206] Oskar Skibski, Szymon Matejczyk, Tomasz P. Michalak, Michael Wooldridge, Makoto Yokoo: k-Coalitional Cooperative Games. AAMAS 2016: 177-185
**Normal-Form Game**

- A game in normal form consists of:
  - Set of players \( N = \{1, \ldots, n\} \)
  - Strategy set \( S \)
  - For each \( i \in N \), utility function \( u_i : S^n \to \mathbb{R} \): if each \( j \in N \) plays the strategy \( s_j \in S \), the utility of player \( i \) is \( u_i(s_1, \ldots, s_n) \)

- Next example created by taking screenshots of
  [http://youtu.be/jILgxeNBK_8](http://youtu.be/jILgxeNBK_8)
Selling ice cream at the beach. One day your cousin Teddy shows up. His ice cream is identical!

You split the beach in half; you set up at 1/4. 50% of the customers buy from you. 50% buy from Teddy.

One day Teddy sets up at the 1/2 point! Now you serve only 37.5%!
THE ICE CREAM WARS

• \( N = \{1,2\} \)
• \( S = [0,1] \)
• \( u_i(s_i, s_j) = \begin{cases} \frac{s_i + s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i + s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases} \)

• To be continued...
THE PRISONER’S DILEMMA

• Two men are charged with a crime
• They are told that:
  ○ If one rats out and the other does not, the rat will be freed, other jailed for nine years
  ○ If both rat out, both will be jailed for six years
• They also know that if neither rats out, both will be jailed for one year
# The Prisoner's Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>-1, -1</td>
<td>-9, 0</td>
</tr>
<tr>
<td>Defect</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
</tbody>
</table>

What would you do?
Understanding the dilemma

• Defection is a dominant strategy
• But the players can do much better by cooperating
• Related to the tragedy of the commons
IN REAL LIFE

• Presidential elections
  o Cooperate = positive ads
  o Defect = negative ads

• Nuclear arms race
  o Cooperate = destroy arsenal
  o Defect = build arsenal

• Climate change
  o Cooperate = curb CO$_2$ emissions
  o Defect = do not curb
On TV

http://youtu.be/S0qjK3TWZE8
**The Professor’s Dilemma**

<table>
<thead>
<tr>
<th>Professor</th>
<th>Class</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Make effort</strong></td>
<td><strong>Listen</strong></td>
<td>$10^6, 10^6$</td>
</tr>
<tr>
<td><strong>Slack off</strong></td>
<td><strong>Listen</strong></td>
<td>0, -10</td>
</tr>
</tbody>
</table>

Dominant strategies?
Nash equilibrium

• Each player’s strategy is a best response to strategies of others.

• Formally, a Nash equilibrium is a vector of strategies $s = (s_1 \ldots, s_n) \in S^n$ such that
  $\forall i \in N, \forall s_i' \in S, u_i(s) \geq u_i(s_i', s_{-i})$
Nash equilibrium

Poll 1: How many Nash equilibria does the Professor’s Dilemma have?

1. 0
2. 1
3. 2 •
4. 3

<table>
<thead>
<tr>
<th>Make effort</th>
<th>Listen</th>
<th>Sleep</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^6, 10^6</td>
<td>-10, 0</td>
<td></td>
</tr>
<tr>
<td>Slack off</td>
<td>0, -10</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

15780 Spring 2017: Lecture 18
NASH EQUILIBRIUM

http://youtu.be/CemLiS15ox8
Russel Crowe was wrong

Hey, Dr. Nash, I think those guys over there are eyeing us. This is like your Nash equilibrium, right? One of them is hot, but we should each flirt with one of her less-desirable friends. Otherwise we risk coming on too strong to the hot one and just driving the group off.

Well, that’s not really the sort of situation I wrote about. Once we’re with the ugly ones, there’s no incentive for one of us not to try to switch to the hot one. It’s not a stable equilibrium.

Crap, forget it. Looks like all three are leaving with one guy.

Darnit, Feynman!
End of the Ice Cream Wars

Day 3 of the ice cream wars...

The PLAN

75%

Teddy sets up south of you!

You go south of Teddy.

Eventually...

Nash Equilibrium
This is why competitors open their stores next to one another!
# Rock-Paper-Scissors

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>P</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>S</td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Nash equilibrium?
**Mixed Strategies**

- A mixed strategy is a probability distribution over (pure) strategies.
- The mixed strategy of player \( i \in N \) is \( x_i \), where
  \[
  x_i(s_i) = \Pr[i \text{ plays } s_i]
  \]
- The utility of player \( i \in N \) is
  \[
  u_i(x_1, \ldots, x_n) = \sum_{(s_1, \ldots, s_n) \in S^n} u_i(s_1, \ldots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)
  \]
**Exercise: Mixed NE**

- **Exercise:** player 1 plays \((\frac{1}{2}, \frac{1}{2}, 0)\), player 2 plays \((0, \frac{1}{2}, \frac{1}{2})\). What is \(u_1\)?
- **Exercise:** Both players play \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). What is \(u_1\)?
**Exercise: Mixed NE**

- **Poll 2: Which is a NE?**
  1. \( \left( \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \right) \)
  2. \( \left( \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( \frac{1}{2}, 0, \frac{1}{2} \right) \right) \)
  3. \( \left( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right) \)
  4. \( \left( \left( \frac{1}{3}, \frac{2}{3}, 0 \right), \left( \frac{2}{3}, 0, \frac{1}{3} \right) \right) \)

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R</strong></td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td><strong>P</strong></td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Any other NE?
NASH’S THEOREM

• Theorem [Nash, 1950]: In any (finite) game there exists at least one (possibly mixed) Nash equilibrium

What about computing a Nash equilibrium?
Does NE make sense?

- Two players, strategies are \{2, \ldots, 100\}
- If both choose the same number, that is what they get
- If one chooses \(s\), the other \(t\), and \(s < t\), the former player gets \(s + 2\), and the latter gets \(s - 2\)
- **Poll 3**: What would you choose?
SUMMARY

• Terminology:
  o Normal-form game
  o Nash equilibrium
  o Mixed strategies

• Nobel-prize-winning ideas:
  o Nash equilibrium 😊