More or Less Review

- A problem $Y$ is **NP-hard** if for every problem $X \in \mathbf{NP}$, $X \leq_p Y$.
- A problem is **NP-complete** if it is both in $\mathbf{NP}$ and NP-hard.
- The goal of an optimization problem is to find the minimum (or maximum) value under some constraints.
- $\text{OPT}(I)$ is the value of the optimal solution to an instance $I$ of an optimization problem.
- We say an algorithm $\mathcal{A}$ for an optimization problem is a factor-$\alpha$ approximation if for all instances $I$ of the problem, $\mathcal{A}$ outputs a solution that is at least as good as $\alpha \cdot \text{OPT}(I)$.

Edge Cover-Up

Let $G = (V, E)$ be a graph. A vertex covering of $G$ is a set $C \subseteq V$ such that for every edge $\{x, y\} \in E$, either $x \in C$ or $y \in C$ (a set of vertices such that every edge is incident to at least one vertex in the set). An independent set in $G$ is a set $S \subseteq V$ such that for any $u, v \in S$, $\{u, v\} \notin E$ (a set of vertices such that no edge connects two vertices in the set). Define the following languages:

- **VERTEX-COVER**: $\{\langle G, k \rangle : G$ is a graph, $k \in \mathbb{N}^+, G$ contain a vertex covering of size $k$}\}
- **IND-SET**: $\{\langle G, k \rangle : G$ is a graph, $k \in \mathbb{N}^+, G$ contains an independent set of size $k$\}\}

Show that $\text{VERTEX-COVER} \leq_p \text{IND-SET}$.

Cut and Dried

We define the Max-Cut problem as follows:

Let $G = (V, E)$ be a graph. Given a coloring of the vertices with 2 colors, we say that an edge $e = \{u, v\}$ is cut if $u$ and $v$ are colored differently. In the Max-Cut problem, the input is a graph $G$, and the output is a coloring of the vertices with 2 colors that maximizes the number of cut edges.

Consider the following approximation algorithm for the Max-Cut problem:

```plaintext
function MaxCutApprox(G = (V, E))
    color all vertices red
    while there exists $v \in V$ s.t. changing $v$'s color increases the number of cut edges do
        change the color of $v$
    end while
    return coloring
end function
```

(a) Show that this algorithm is poly-time.

(b) Prove that this algorithm is a $\frac{1}{2}$-approximation for Max-Cut.

(c) Show that this algorithm is not a $(\frac{1}{2} + \varepsilon)$-approximation algorithm for Max-Cut for any $\varepsilon > 0$. 
Gotta Catch a Lot of 'Em

Consider a set of Pokémon and a set of $m$ trainers each having a subset of these Pokémon. Given an integer $k$, the problem is to pick $k$ trainers in a way that maximizes the number of distinct Pokémon owned among them. This problem will show that there exists a poly-time $(1 - 1/e)$-approximation by considering the following greedy algorithm:

$$
\text{function } \text{PokémonApprox}(\langle S_1, S_2, \ldots, S_m \rangle, k) \\
T \leftarrow \emptyset \quad \triangleright \text{keeps track of trainers we have already picked} \\
U \leftarrow \emptyset \quad \triangleright \text{keeps track of which Pokémon we have already}
$$

$$
\text{for } 1 \leq i \leq k \text{ do} \\
\quad j \leftarrow \arg \max_{j} |S_j - U| \quad \triangleright \text{pick the trainer } j \text{ with the most new Pokémon} \\
\quad T \leftarrow T \cup \{ j \} \\
\quad U \leftarrow U \cup S_j
$$

end for

return $T$

end function

(a) Prove that the algorithm runs in poly-time.

(b) Let $T^*$ denote the optimum solution, and let $U^* = \bigcup_{j \in T^*} S_j$. Further, define $U_i$ to be the set $U$ in the algorithm after the $i$-th iteration of the loop. Prove that $|U^*| - |U_i| \leq (1 - \frac{1}{k})^i |U^*|$.

(c) Using the inequality $1 + x \leq e^x$, deduce that this algorithm is a $(1 - \frac{1}{e})$-approximation.

(Extra) Looping Around

Show that the HALTS is NP-hard.

(Bonus) Hard Cut

On the previous page, we defined Max-Cut as an optimization problem. We can also define the decision version MAX-CUT as follows:

**MAX-CUT**: $\{ \langle G, k \rangle : G$’s vertices may be colored with two colors in a way that cuts at least $k$ edges $\}$. 

Prove that MAX-CUT is NP-hard. This is slightly difficult; try reducing from IND-SET.