Training Manual

- **Deterministic Finite Automaton (DFA):** A DFA $M$ is a machine that reads a finite input one character at a time in one pass, transitions from state to state, and ultimately accepts or rejects. Formally, $M$ is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where $Q$ is the finite set of states, $\Sigma$ is the finite alphabet, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, $q_0 \in Q$ is the starting state, and $F \subseteq Q$ is the set of accepting states.

- **Regular language:** A language $L$ is regular if $L = L(M)$ for some DFA $M$ ($M$ recognizes $L$).

  We have shown that if $L_1$ and $L_2$ are both regular languages over $\Sigma^*$, for some fixed $\Sigma$, then the following are all regular.
  
  - $L_1$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$ (the concatenation of two regular languages)

- **Turing Machine (TM):** A TM $M$ is a machine that can read and write to an infinite tape containing the input, transition from state to state, and ultimately accept, reject, or loop infinitely. Formally, $M$ is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$, where:
  
  - $Q$ is the finite set of states,
  - $\Sigma$ is the finite input alphabet with $\sqcup \notin \Sigma$,
  - $\Gamma$ is the finite tape alphabet with $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$,
  - $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
  - $q_0 \in Q$ is the starting state,
  - $q_{acc} \in Q$ is the accepting state,
  - and $q_{rej} \in Q, q_{rej} \neq q_{acc}$ is the rejecting state.

- **Decider TM:** A TM $M$ is a decider if it halts on all inputs.

- **Decidable language:** A language $L$ is decidable (or computable) if $L = L(M)$ for some decider TM $M$. 
Drawing DFAs

(a) Draw a DFA recognizing the language $L$ over \{a, b\} where $L$ is the set of strings that begin and end with the same character.

(b) Draw a DFA that recognizes the language

\[ L = \{ x : x \text{ has an even number of 1s and an odd number of 0s} \} \]

over the alphabet $\Sigma = \{0, 1\}$.

A Santa Lived As a Devil At NASA!
Show that, if $|\Sigma| > 1$, then

\[ \text{PAL} = \{ x \mid x \in \Sigma^* \text{ and } x = x^R \} \]

is an irregular language, where $x^R$ denotes the reverse of the string $x$.

Reversing Regular Languages
If $A$ is a regular language over $\Sigma$, then show that $A^R$ (the reversal of $A$) is regular by providing a DFA for it.

Balance in All Things
Construct a TM that decides the language $L = \{ x : \text{the parentheses in } x \text{ are balanced} \}$ over the alphabet $\Sigma = \{(,)\}$.

Multiple Multiples (Extra Problem)
Let $\Sigma = \{0, 1\}$. For each $n \geq 1$, define

\[ C_n = \{ x \in \Sigma^* \mid x \text{ is a binary number that is a multiple of } n \} \]

Show that $C_n$ is regular for all $n$. 