15-150

Principles of Functional Programming

Lecture 4

January 25, 2024

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Tail Recursion

More about

Lists
Structural Induction

```
(* length : int list → int
   REQUIRES: true
   ENSURES:  length(L) returns the
             number of elements in L.
*)

fun length ([]: int list): int = 0
  | length (x::xs) = 1 + length(xs)
```

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(* length : int list → int
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            number of elements in L.
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fun length ([]: int list): int = 0
  | length (x::xs) = 1+ length(xs)
```

```
  length [4,7,9,2]
⟹ 1 + length [7,9,2]
```

Why?

```
(* length : int list → int
   REQUIRES: true
   ENSURES: length(L) returns the
            number of elements in L.
*)

fun length ([]: int list): int = 0
  | length (x::xs) = 1+ length(xs)
```

length [4,7,9,2]

$\Longrightarrow$ 1 + length [7,9,2]

Why?

Because [4,7,9,2] means 4::[7,9,2]

and so length [4,7,9,2]

$\Longrightarrow$ [..., 4/x, [7,9,2]/xs] 1+ length(xs)

$\Longrightarrow$ 1 + length [7,9,2]

(... means the environment when length was defined) length [4,7,9,2]

$\Longrightarrow$ 1 + length [7, 9, 2]

$\Longrightarrow$ 1 + (1 + length [9, 2])

$\Longrightarrow$ 1 + (1 + (1 + length [2]))

$\Longrightarrow$ 1 + (1 + (1 + (1 + length [])))

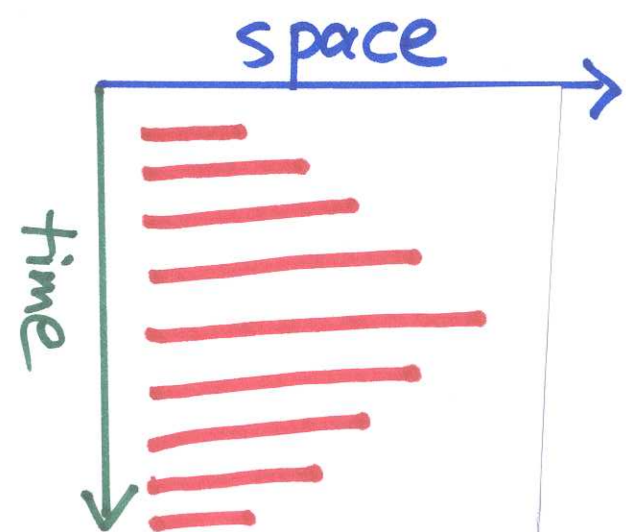
$\Longrightarrow$ 1 + (1 + (1 + (1 + 0)))

$\Longrightarrow$ 1 + (1 + (1 + 1))

$\Longrightarrow$ 1 + (1 + 2)

$\Longrightarrow$ 1 + 3

$\Longrightarrow$ 4



accumulator

(* tlength: int list * int → int

REQUIRES: true

ENSURES: tlength(L, acc) $\cong$ (length L) + acc

*)

```
(* tlength : int list * int -> int
   REQUIRES: true
   ENSURES:  tlength(L, acc)
                  ≅
             (length L) + acc
*)

fun tlength ([] : int list, acc : int) : int = ?
```

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(* tlength : int list * int -> int
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fun tlength ([] : int list, acc : int) : int = acc
  | tlength (x::xs, acc) =

            ?
```

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(* tlength : int list * int -> int
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fun tlength ([] : int list, acc : int) : int = acc
  | tlength (x::xs, acc) =
      tlength (xs, 1+acc)
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(* tlength : int list * int -> int
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fun tlength ([] : int list, acc : int) : int = acc
  | tlength (x::xs, acc) =
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```

tail call

---

tlength is tail recursive

Definition

A function is tail recursive if it is recursive and if it performs no computations after calling itself recursively.

Such recursive calls are said to be tail calls

(as in: "tail" meaning "at the end").

If the body of a function contains multiple locations at which a recursive call occurs, then every recursive call must be a tail call for the function to be tail recursive.

Now implement a length function based on tlength:

```
(* leng : int list → int
   REQUIRES & ENSURES as for length
*)

fun leng (L: int list) : int =

???
```

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fun leng (L: int list) : int =

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tlength (L, acc) $\cong$ (length L) + acc

Now implement a length function based on tlength:

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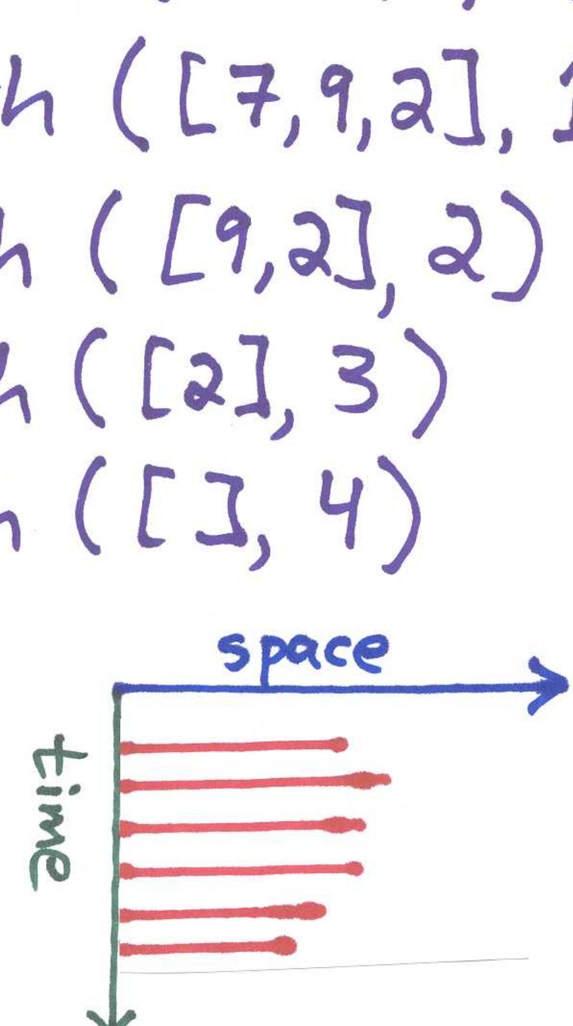
leng [4,7,9,2]

$\Longrightarrow$ tlength([4,7,9,2], 0)

```
fun tlength ([]:int list, acc:int):int = acc
  | tlength (x::xs, acc) =
      tlength (xs, 1+acc)
```

```
   leng [4,7,9,2]
⟹ tlength ([4,7,9,2], 0)
⟹ tlength ([7,9,2], 1)
⟹ tlength ([9,2], 2)
⟹ tlength ([2], 3)
⟹ tlength ([], 4)
⟹ 4
```



Theorem

For all values L : int list and acc : int,

$$\text{tlength}(L, acc) \cong (\text{length } L) + acc.$$

During lecture:

We proved the theorem using structural induction.
See online code file for details.

```
(* append : int list * int list -> int list
     REQUIRES: true
     ENSURES:
       append (X, Y) returns a list
       consisting of the elements of
       X followed by the elements of
       Y, preserving order.
   Example:  append ([3,4], [1,3,10])
               ==> [3,4,1,3,10]
*)

fun append ([]: int list, Y : int list) : int list = Y
  | append (x::xs, Y) = x :: append (xs, Y)
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```
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               ==> [3,4,1,3,10]
*)

fun append ([]: int list, Y : int list): int list = Y
  | append (x::xs, Y) = x :: append (xs, Y)
```

---

append (X, Y) has time complexity $O(|X|)$.

```
fun append ([]: int list, Y : int list) : int list = Y
  | append (x::xs, Y) = x :: append (xs, Y)
```

```
append ([1,2], [5,~6,7])
⟹ 1::append ([2], [5,~6,7])
⟹ 1:: (2 :: append ([ ], [5,~6,7]))
⟹ 1:: (2 :: [5,~6,7])
⟹ 1:: [2,5,~6,7]
⟹ [1,2,5,~6,7]
```

append is predefined in SML as the right-associative infix operator @.

So $[1,2] \mathbin{@} [3,4] \mathbin{@} [6,9,10]$

means

$$[1,2] \mathbin{@} ([3,4] \mathbin{@} [6,9,10])$$

$$\Longrightarrow [1,2] \mathbin{@} [3,4,6,9,10]$$

$$\Longrightarrow [1,2,3,4,6,9,10]$$

(* rev : int list → int list

REQUIRES: true

ENSURES: rev L returns a list consisting of the elements of L in reverse order.

Example: rev [7,9,2] ⟹ [2,9,7].

*)

```
(* rev : int list → int list
   REQUIRES: true
   ENSURES:  rev L returns a list
        consisting of the elements of L
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   Example:  rev [7,9,2] ⟹ [2,9,7].
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fun rev ([]:int list):int list = ?
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fun rev ([]:int list):int list = []
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   Example: rev [7,9,2] ⟹ [2,9,7].
*)

fun rev ([]:int list):int list = []
  | rev (x::xs) = ?
```

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   Example:  rev [7,9,2] ==> [2,9,7].
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fun rev ([]:int list):int list = []
  | rev (x::xs) = (rev xs) @ [x]
```

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What is the time complexity?

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(* rev : int list → int list
   REQUIRES: true
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   Example: rev [7,9,2] ⟹ [2,9,7].
*)
```

```
fun rev ([]:int list):int list = []
  | rev (x::xs) = (rev xs) @ [x]
```

What is the time complexity?

$O(n^2)$,

with n the number of elements in the list.

```
  rev [1,2,3,4]
⟹ (rev [2,3,4]) @ [1]
⟹ ((rev [3,4]) @ [2]) @ [1]
⟹ (((rev [4]) @ [3]) @ [2]) @ [1]
⟹ ((((rev [ ]) @ [4]) @ [3]) @ [2]) @ [1]
⟹ ((([ ] @ [4]) @ [3]) @ [2]) @ [1]
⟹ (([4] @ [3]) @ [2]) @ [1]
⟹ ((4 :: [ ] @ [3]) @ [2]) @ [1]
⟹ ((4 :: [3]) @ [2]) @ [1]
⟹ ([4,3] @ [2]) @ [1]
⟹ (4 :: [3] @ [2]) @ [1]
⟹ (4 :: (3 :: [ ] @ [2])) @ [1]
⟹ (4 :: (3 :: [2])) @ [1]
⟹ (4 :: [3,2]) @ [1]
⟹ [4,3,2] @ [1]
⟹ 4 :: [3,2] @ [1]
⟹ 4 :: (3 :: [2] @ [1])
⟹ 4 :: (3 :: (2 :: [ ] @ [1]))
⟹ 4 :: (3 :: (2 :: [1]))
⟹ 4 :: (3 :: [2,1])
⟹ 4 :: [3,2,1]
⟹ [4,3,2,1]
```

Finally, [4,3,2,1]

```
(* trev : int list * int list -> int list
   REQUIRES: true
   ENSURES:   trev(L,acc)
                  ≅
              (rev L)@ acc
*)
```

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(* trev : int list * int list -> int list
   REQUIRES: true
   ENSURES:   trev(L,acc)
                  ≅
              (rev L)@ acc
*)

fun trev ([]:int list, acc:int list) : int list =
                         ?
```

```
(* trev : int list * int list -> int list
   REQUIRES: true
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fun trev ([]:int list, acc:int list) : int list =
                 acc
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   REQUIRES: true
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*)

fun trev ([]:int list, acc:int list): int list =
                      acc

  | trev (x::xs, acc) =    ?
```

```
(* trev : int list * int list -> int list
   REQUIRES: true
   ENSURES:  trev(L,acc)
                 ≅
             (rev L)@ acc
*)

fun trev ([]:int list, acc:int list):int list =
                acc
  | trev (x::xs, acc) = trev(xs, x::acc)
```

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(* trev : int list * int list -> int list
   REQUIRES: true
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fun trev ([]:int list, acc : int list) : int list =
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What is the time complexity?

```
(* trev : int list * int list -> int list
   REQUIRES: true
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fun trev ([]:int list, acc:int list): int list =
                  acc

  | trev (x::xs, acc) = trev (xs, x::acc)
```

What is the time complexity?

$O(n)$,

with n the number of elements in the first list.

trev([1,2,3,4], [ ])

$\Longrightarrow$ trev([2,3,4], [1])

$\Longrightarrow$ trev([3,4], [2,1])

$\Longrightarrow$ trev([4], [3,2,1])

$\Longrightarrow$ trev([ ], [4,3,2,1])

$\Longrightarrow$ [4,3,2,1]

Can now implement list reversal more efficiently:

```
(* reverse : int list → int list *)

fun reverse (L:int list):int list =
```

???

$$\text{trev}\,(L, acc) \cong (\text{rev}\ L) @ acc$$

Can now implement list reversal more efficiently:

```
(* reverse : int list → int list *)

fun reverse (L:int list):int list =

        trev (L, [])
```

$\text{trev}(L, acc) \cong (\text{rev } L) @ acc$

Theorem

For all values L : int list and acc : int list,

$$\text{trev}(L, acc) \cong (\text{rev } L) @ acc.$$

During lecture:

We proved the theorem using structural induction.
See online notes for details.

That is all.

Have a good weekend.

See you Tuesday.