Number Games: Its raining outside and Alfonso and Bernadette are bored. Alfonso suggests the following games:

(a) Two players alternatively erase some 9 numbers from the sequence 1,2,...,101 until only two remain. The player that starts wins $x - 54$ dollars from the player that plays second. Here x is the difference between the remaining two numbers. Would you rather be the first or the second player?

(b) Two players alternatively erase one number from the sequence 1,2,...,27 until only two numbers remain. The first player wins if the sum of these numbers is divisible by 5; otherwise the second player wins. Who has a winning strategy?

Solution
(a) The first player has a winning strategy. At his first turn the first player can delete numbers 47 - 55. The remaining numbers can now be paired up into 46 pairs (1-56,2-57,....,46-101). Then at every pair of turns (second player, first player) the first player has to make sure that exactly 9 of these pairs get erased. At the end of play there will be one pair left and the first player will win one dollar.
(b) The first player A has a winning strategy. The state of the game at any time can be described by a sequence $x_{-2}, x_{-1}, x_0.x_1, x_2$ where x_j is the number of integers remaining that are equal to $j \mod 5$. If A can arrange things so that the other player B faces a position where $x_j = x_{-j}$ for $j = 1, 2$ and x_0 is even then player A will win. Call such a position a *balanced position.* If B takes a number which is $k \mod 5$ then in the next round A will take a number which is $-k \mod 5$ and once again there will be a balanced position. At the end, when there are 2 numbers left and they form a balanced position, A will have won.
A begins by taking number 1. After this we almost have a balanced position, except that there is an extra 0 and 2. A plays as if in a balanced position up until B first takes a 0 or 2, in which case A will respond by taking the other choice. After this it will be a balanced position.
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