The case of the quarrelsome quartets.

The Ruritanian state choir has 120 members. Some pairs of members are friends. But, a friend of Carla's friend is not necessarily Carla's friend. In a tradition dating back to a time when the choir was somewhat smaller, every set of four members from the choir are required to prepare a song together. The choirmaster has noticed that the most quarrelsome quartets are those where two of the members are friends, but no other friendships exist within the quartet. She has also noticed that the present number of quarelsome quartets is as large as it possibly can be. How many quarrelsome quartets are there?

Solution

We view this as a problem in graph theory. For each graph, G, with 120 vertices, we count the number of four element subsets of its vertices with the property that they include exactly one edge. Denote this quantity by $t(G)$. We seek to find the maximum value M of $t(G)$.

For a graph H and two vertices $c, d \in H$ define $H_{c \leftarrow d}$ to be the graph formed from H by removing all the edges of the form cx for $x \neq d$, and replacing them by edges cy for each y such that dy is an edge. Note, we do not change whether or not cd is an edge. It helps to think of this operation as "cloning d onto c".

Consider any graph G, and any two vertices $a, b \in G$ such that ab is an edge. Among the subgraphs of G contributing to $t(G)$ there are four types: those containing neither a nor b, those containing a but not b, those containing b but not a, and those containing both a and b.

The subgraphs of the first type are unchanged in $G_{a \to b}$ and $G_{b \leftarrow a}$. $G_{b \leftarrow a}$ contains twice as many of the second type (counting the "cloned" versions separately) as G does, and $G_{a \to b}$ twice as many of the third type. Each contains at least as many of the fourth type (and possibly more). This establishes that:

$$t(G_{a \leftarrow b}) + t(G_{b \leftarrow a}) \geq 2t(G).$$

In particular, if $t(G) = M$, we must have $t(G_{a \leftarrow b}) = t(G_{b \leftarrow a}) = M$. In, say, the second of these the pair of elements a and b have the same neighbours. By successively cloning onto a we can ensure that all of a's neighbours have the same neighbours as a does – that is, they form a clique independent of the rest of the graph. By a sequence of such transformations we can produce a graph H with $t(H) = M$ that has the additional property that the vertices of H all belong to independent cliques. Let the sizes of these cliques be $x_1, x_2, \ldots, x_{120}$ where each $x_i \geq 0$ (and we allow $x_i = 0$). Now:

$$t(H) = \sum_{i=1}^{120} \binom{x_i}{2} \left(\sum_{j<k, j \neq i, k \neq i} x_j x_k \right) \tag{1}$$

Consider any two of the x_i, and denote their values by x and y. We examine the expression above as a function of x and y alone, i.e. we hold the other values fixed. It has the form:

$$C_1 \left(\binom{x}{2} + \binom{y}{2} \right) + C_2(x+y) + C_3 \left(\binom{x}{2} y + \binom{y}{2} x \right)$$

where C_1, C_2, and C_3 do not depend on x or y. Substituting $y = S - x$ (where S is the, fixed, sum of x and y) we get an expression which is quadratic in x and symmetric about $S/2$[1]. Its maximum on the interval $[0, S]$ therefore occurs either at $x = 0$ (and $x = S$) or at $x = S/2$. Restricting to integer values of x, the maximum occurs at one of the points above (if S is even), or at $x = (S \pm 1)/2$ if S is odd.

In terms of the original problem this implies that all the non-zero x_i differ from one another by at most 1.

Thus the values of the x_i are determined (as a multiset) by the number of non-zero x_i. Now let $T(n)$ denote the value of $t(H)$ when n of the x_i are non-zero. We see immediately that:

$$\begin{aligned}
T(1) = T(2) &= 0 \\
T(3) &= 3\binom{40}{2}(40)(40) \\
T(4) &= 4\binom{30}{2}\binom{3}{2}(30)(30) \\
T(5) &= 5\binom{24}{2}\binom{4}{2}(24)(24) \\
T(6) &= 6\binom{20}{2}\binom{5}{2}(20)(20).
\end{aligned}$$

Of these, it is easy to check that $T(5)$ is the largest. For all n:

$$\begin{aligned}
T(n) &\leq n\frac{(120/n)((120/n) - 1)}{2}\binom{n-1}{2}(120/n)^2 \\
&< \frac{120^4(n-1)(n-2)}{n^3}
\end{aligned}$$

since this expression represents the maximum of equation (1) over the region where the x_i are non-negative n of the x_i are non-zero, and their sum is 600. The final expression above is decreasing as a function of n and smaller than $T(5)$ for $n \geq 7$.

Thanks to Michael Albert for providing the problem and its solution. Nathan Abraham worked out the maximum number too.

Notes

By increasing the number of members in the choir one can ensure that the estimate provided works for $n \geq 6$.

A slightly easier problem deals with "troublesome trios". However, in that form the configuration consists of two equal sized cliques – which is similar to other problems of this type. In particular, the methods of proof in Tarjan's theorem (on the maximum number of edges in a triangle free graph) may suggest themselves. This may not be a bad thing. On the other hand one of the most

[1] It appears to be cubic but a cubic cannot be symmetrical about a point!

attractive aspects of the present version is that the configuration which gives the largest number of quarrelsome quartets, has five groups!

One nice feature of the present version is that if you try to carry out the estimates by analytic methods, you can get yourself into a terrible muddle.

The problem has its genesis in the investigations reported in the paper: M.H. Albert, M.D. Atkinson, C.C. Handley, D.A. Holton and W. Stromquist, *On Packing Densities of Permutations*, Electronic J. Combinatorics 9(1):R5, 2002. However, the only general argument from that paper applicable to this form would be the one which gives the fact that a maximising graph may be taken to consist of independent cliques.