Here comes Santa

Santa has been working hard all year and is now waiting at the bottom left corner $(0,0)$ of Gridville, sacks loaded with presents, ready to deliver them to the children waiting at the top edge $\{(x,n) : \ 0 \le x \le n\}$. Santa laid off too many elves last year and now things are running late. There are only $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ days to Xmas.The Grinch see his chance to mess things up. Santa's reindeer love Magic Reindeer Food (made from sugar, oatmeal and glitter) and the Grinch sprinkles $F_{i,j}$ days worth of food on grid square (i,j) for $1 \le i,j \le n$. Magic Reindeer Food is in short supply and the Grinch can only afford to spread one days supply on each row i.e. $\sum_{j=1} F_{i,j} = 1$ for $1 \le j \le n$. If Santa's journey takes him to square (i,j) then he will have to spend $F_{i,j}$ of a day there while the reindeer's eat the food that the Grinch has left.
Santa starts at $(1,1)$ and waits $F_{1,1}$ days and then moves to $(2,1)$ or $(2,2)$. In general, if Santa is at (i,j), then after waiting $F_{i,j}$ of a day, he moves to one of $(i+1,j-1)$, $(i+1,j)$, $(i+1,j+1)$. (Of course if $j=1$ or n only 2 of these moves are allowed).
Santa can make all his deliveries after reaching a square (n,j) (and waiting $F_{n,j}$) because he can mount a non-deterministic delivery service from this time on. Can he make it in time, or will the Grinch ruin Xmas?

Solution: Let $D_{i,j}$ be the shortest distance from $(0,0)$ to (i,j) (including the wait at (i,j)). Let $S_i = \sum_{j=1}^n D_{i,j}$.
Claim: $S_{i+1} \le 1 + S_i + \frac{1}{i} S_i$ for $1 \le i < n$.
Proof of Claim Fix i and suppose that $D_{i,j^*} = \min D_{i,j}$. Define the following map $f : [i+1] \to [i]$.

$$f(j) = \begin{cases} j & j \le j^* \\ j-1 & j > j^* \end{cases}.$$

Now we have

$$D_{i+1,j} \le D_{i,f(j)} + F_{i+1,j}$$

and so by summing over j

$$S_{i+1} \le 1 + S_i + D_{i,f(j^*)} \le 1 + S_i + \frac{1}{i} S_i.$$

□

From the claim we get that

$$\frac{S_{i+1}}{i+1} \leq \frac{1}{i} + \frac{S_i}{i}$$

from which we deduce that

$$\frac{1}{n} S_n \leq H_n.$$

The result follows now from

$$\min_j D_{n,j} \leq \frac{1}{n} S_n \leq H_n.$$

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