Vacancy Puzzle: Solution

Let us change notation. Suppose that we write $a(c)$ for the priority of the candidate $c \in C$. We show that Arctan can win if and only if in the initial position

$$\phi(C) = \sum_{c \in C} 2^{-a(c)} < 1. \tag{1}$$

Ww will prove this by induction on the number of steps left in the process. At the end of the process either $C = \emptyset$ and $\phi(C) = 0$ and Arctan wins or there are some members of C with $a(c) = 0$ and $\phi(C) \geq 1$ and then Arcsin wins.

Now consider a general step. Suppose that ϕ' refers to the new value of ϕ after Arctan has made his selection. Then we have

$$\phi(C) = \frac{1}{2}\phi'(S) + \frac{1}{2}\phi'(C \setminus S). \tag{2}$$

So if $\phi(C) < 1$ then $\min\{\phi'(S), \phi'(C \setminus S)\} < 1$ and so Arctan can ensure that $\phi' < 1$ by choosing S or $C \setminus S$ and then he will win, by induction.

If $\phi(C) \geq 1$ then we can use the following lemma.

Lemma 1 *Let $x_1 \geq x_2 \geq \cdots \geq x_r$, $r \geq 2$, all be negative powers of 2 with sum $x_1 + x_2 + \cdots + x_r \geq 1$. Then there exists a partition of the x_i into two groups so that each group sums to at least one half.*

It follows that if $\phi(C) \geq 1$ then Arcsin can find a set S such that $\min\{\phi'(S), \phi'(C \setminus S)\} \geq 1$ and then induction implies that Arcsin will win.

Proof of Lemma 1 We can assume without loss of generality that $x_1 + x_2 + \cdots + x_r = 1$. Either $r = 2$ and the result is trivial or $r \geq 3$ and $x_1 + x_2 + \cdots + x_r - 1 \geq x_r$ and we can place x_r arbitrarily. Here we use the fact that the x_i's are negative powers of 2.

We use induction on r. Now we must have $x_{r-1} = x_r$, again because the x_i's are negative powers of 2. If $r = 2$ then $x_{r-1} = x_r = \frac{1}{2}$ and the result is trivial. otherwise, we replace x_{r-1}, x_r by $x_{r-1} + x_r$ and use induction. □

This problem was called the "Tenure Game" in the paper by Joel Spencer [1].

References

[1] J. Spencer, Randomization, Derandomization, and Antirandomization: Three games, *Theoretical Computer Science* 131 (1994), 415-430