Take the last chip: solution

Let us generalise the game in the following way. There are 2 players A and B and A goes first. We have a non-decreasing function f from $N \to N$ where $N = \{1, 2, \ldots\}$ is the set of natural numbers. At the first move A takes any number less than h from the pile, where h is the size of the initial pile. Then on a subsequent move, if a player takes n chips then the next player is constrained to take at most $f(n)$ chips. Thus the puzzle considered the cases $f(n) = n$ and $f(n) = 2n$.

There is a set $\mathcal{H} = \{H_1 = 1 < H_2 < \ldots\}$ of initial pile sizes for which the first player will lose, assuming that the second player plays optimally. Also, if the initial pile size $h \notin \mathcal{H}$ then the first player has a winning strategy. The following theorem is taken from Zieve [2] and is attributed there to Epp and Ferguson [1].

Theorem
If $f(H_j) \geq H_j$ then $H_{j+1} = H_j + H_\ell$ where

$$H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}.$$

Furthermore, if $f(H_j) < H_j$ then the sequence of losing positions is finite and ends with H_j.

Before proving the theorem we observe that the theorem implies

$$f(n) = n \quad \textit{implies} \quad \mathcal{H} = \{1, 2, 4, \ldots, 2^k, \ldots,\}$$
$$f(n) = 2n \quad \textit{implies} \quad \mathcal{H} = \{1, 2, 3, \ldots, F_k, \ldots,\}$$

where the F_k are the Fibonacci numbers.

Proof of theorem Assume that $f(H_j) \geq H_j$; then $H_\ell = \min_{i \leq j}\{H_i \mid f(H_i) \geq H_j\}$ exists. For any losing position $H_i < H_\ell$, we have $f(H_i) < H_j$, so from an initial pile of size $H_j + H_i$, Player A can remove H_i chips and win, since this leaves B with a pile of size H_j from which he/she cannot remove all chips.

Now let $x < H_\ell$ be a winning position. Given a pile of size $H_j + x$, Player A can employ a winning strategy for a pile of size x whose final move involves y chips, where $f(y) < H_j$; this again leaves Player B with a pile of size H_j from which he/she cannot remove all chips. (Player A can always arrange for y to satisfy this property because if the last move y of a winning strategy for x satisfies $f(y) \geq H_j$, then $y < H_\ell$ cannot be a losing posittion – from definition of H_ℓ – and consideration of a winning strategy for y leads to a smaller final move).

Finally, from a pile of size $H_j + H_\ell$, if Player A takes at least H_ℓ chips then Player B takes the rest and wins. If Player A takes less than H_ℓ then we fall into the preceding paragraph's situation with the roles reversed. This proves the first statement of the theorem.

If $f(H_j) < H_j$, suppose we had $H_{j+1} = H_j + x$ for some $x > 0$. As above, x cannot be any H_i, since then Player A wins from $H_j + H_i$ by removing H_i chips, because $f(H_i) \leq f(H_j) < H_j$. Now since $x < H_{j=1}$, x must be a winning position. Thus Player A can win from $H_j + x$ by employing a winning strategy for x whose final move is y, where $f(y) < H_j$. Thus H_{j+1} is not a losing position – contradiction, i.e. there is no H_{j+1}. □

References

[1] R.J. Epp and T.S. Ferguson, A note on Take-away Games, *Fibonacci Quarterly* 18 (1980) 300-303.

[2] M. Zieve, Take-away Games, In *Games of no chance*, R.J. Nowakowski Editor, MSRI Publications 29 (1994) 351-362.