THE LIGHTS OF THE ROUND TABLE

King Arthur is preparing for a meeting of the round table. The seats at the round table are numbered 1 through n. Each seat has a reading lamp and when Arthur goes into the round table room he finds that someone has switched off some of the reading lamps. Being a king, Arthur cannot simply switch on those lights that are currently off. He has to tell somebody else to do it.

So Arthur has to write down a list of numbers of seats and then get a servant to flip the switch on each light that is listed. The next meeting is to discuss the banishment of Merlin and the meeting can only start when all of the lights are turned on. Using a crystal ball Merlin can see the list of numbers that the king has written and is able to rotate the table before the servant gets there.

So for example, if n=100 and the king asks for 3,28,97 to be flipped, then if Merlin rotates the table by 10 places, the servant will in fact flip the switch on lamps that were in positions 93,18,87 and so may flip the switch on a light that is already on. The servant is not allowed to use common-sense. Disobeying the kings instructions can be hazardous to ones health.

SOLUTION

If n is a power of 2 then King Arthur can defeat Merlin regardless of the initial setting of the lights. If n is not a power of 2 then there are initial settings of the lights for which Merlin can postpone the meeting indefinitely.

Proof

1 $n = 2^k$

First suppose thet $n = 2^k$. We give 2 proofs that Arthur can succeeed.

1.1 Inductive proof

The base case $k = 0$ is trivial.

From now on let us equate 1 with on and 0 with off and think of the light setting as a bit string $b_0, b_1, \ldots, b_{n-1}$. Assume inductively that Arthur has a light setting strategy L_{k-1} when $n = 2^{k-1}$. In the case $n = 2^k$ let $X_i = X_{i+2^{k-1}} = \{i, i + 2^{k-1}\}$ for $i = 0, 1, \ldots, 2^{k-1} - 1$. We think of X_i as a "superlight". It is on if $b_i = b_{i+2^{k-1}}$ and off otherwise. Arthur always gives the servant a set of lights from $\{0, 1, \ldots, 2^{k-1} - 1\}$. Then if Arthur asks for j to be flipped on and Merlin gets $m + j$ flipped instead, this will be analogous to flipping X_{m+j}, since only one member of an X_i can be flipped in one round.

So, after n_{k-1} rounds, say, $b_i = b_{i+2^{k-1}}$ for $i = 0, 1, \ldots, 2^{k-1} - 1$. From now on Arthur only issues orders which consist of a collection of X_i's and he uses L_{k-1} to get both lights in each X_i switched on. Thus Arthur succeeds in $2n_{k-1}$ rounds i.e. $n_k = 2n_{k-1} = \cdots = n$.

1.2 Algebraic Proof

The above algorithm requires more mathematical sophistication than can be expected from a medieval monarch. What if Arthur just selects the lights which are currently on and asks for them to be switched off. If successful, Arthur will get all the lights switched off and then can ask for them all to be switched on next round. Merlin will be helpless.

Let $b(x)$ denote the **state** polynomial $b_0+b_1x+\cdots+b_{n-1}x^{n-1}$. Suppose now that Arthur makes the selection given in the previous paragraph and Merlin rotates the table by j positions. Then

$$b(x) \leftarrow b(x)(1+x^j)$$

where all calculations are done modulo 2 i.e. over GF_2.

So, if the table starts with state polynomial $t(x)$ then after mC rounds the state polynomial will be

$$t(x)\prod_{t=1}^{m}(1+x^{j_t})$$

for some $0 \leq j_1, j_2, \ldots, j_m \leq n-1$.

Now suppose that $m \geq n(n-1)+1$. By the pigeon-hole principle, some i appears at least n times in $\{j_1, j_2, \ldots, j_m\}$.

Claim: $(1+x^i)^n = 0\,\forall i$.

Given the claim we see that after at most $n(n-1)+1$ rounds all light s will have been switched off and Arthur succeeds in the next round.

Proof of Claim

$$(1+x^i)^2 = 1+x^i+x^i+x^{2i} = 1+x^{2i}.$$

Repeating,

$$(1+x^i)^{2^p} = 1+x^{2^p i}$$

and if $p=k$

$$(1+x^i)^n = 1+x^{ni} = 1+1 = 0.$$

□

2 $n \neq 2^k$

Now suppose that n is not a power of 2. As a warm-up, assume that n is odd. Assume initially that

$$\mathbf{b} \notin \{0^n, 1^n\}. \tag{1}$$

We claim that Merlin can maintain (1) indefinitely. We now think of Arthur's choice of lights as a pattern $\mathbf{p} = (p_0, p_1, \ldots, p_{n-1})$ of 0's and 1's. If $\mathbf{p} \in \{0^n, 1^n\}$ then Merlin does not have to rotate the table in order to maintain (1). Otherwise we know that there exists i such that $b_i = 0, b_{i+1} = 1$ ands a j such that $p_j = p_{j+1}$. Now Merlin rotates the table so that i is aligned with j and then the new state $\mathbf{b}$' satisfies $b'_i = 1, b'_{i+1} = 0$ when $p_j = p_{j+1} = 1$ and $b'_i = 0, b'_{i+1} = 1$ when $p_j = p_{j+1} = 0$ which implies (1) for $\mathbf{b}$'.

Now consider the general case. $n = 2^k m$ where m is odd. We partition $N = \{0, 1, \ldots, n-1\}$ into 2^k classes $S_0, S_1, \ldots, S_{2^k-1}$ where $S_i = \{j \in N : n \mod 2^k = j\}$. Let $\mathbf{b}^{(0)}, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(2^k-1)}$ be the associated bit vectors. Now we replace (ref1) by

$$\exists i : \ \mathbf{b}^{(i)} \notin \{0^m, 1^m\}. \tag{2}$$

Now Merlin can choose i satrisfying (2)) and rotate the table to maintain (2) for this i. Let $\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(2^k-1)}$ correspond to Arthur's pattern. Merlin can restrict himself to rotations which are multiples of 2^k. In this way $\mathbf{b}^{(i)}$ is aligned with some rotation of $\mathbf{p}^{(i)}$ and he can use the strategy for n odd (with n replaced by m).

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