Turn on the lights

Toad Hall at OMG University is illuminated by a 4×4 array of powerful lights. There is a corresponding bank of 4×4 switches that are suppose to turn each light off and on. All the lights are currently off and a porter has been sent to turn them on. This should be simple, but the electrics at Toad Hall were recently re-wired by the Acme Electric Company. They are cheap, but they hire ex-comedians. Now when a switch is flipped, all neighboring switches are inadvertently switched too. So, for example, if the current situation is as in the first array and we flip switch (2,2) then the situation becomes as in the second array. You can interpret 0 as off and 1 as on.

To clarify possible confusions, if the switch corresponds to a boundary light, then there will be fewer than four neighbors. Also, by neighbr, we mean horizontal or vertical neighbor, not diagonal.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Is it possible to get all of the lights on?

Now imagine the same problem at Imperial Hall where there is an array of 100×100 lights!

Solution: We begin with a solution to the specific 4×4 problem. Pressing the switches below will turn on all of the lights: Thanks to Eric Shrader for pointing this one out.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now we prove a theorem by Sutner [1]: Let $G = (V, E)$ be an arbitrary graph. Suppose that each vertex carries a label $\ell(v) = 0$ or 1. Suppose that if $v \in V$, the transformation $T(v)$ flips the values at v and all of its neighbors. (Flip from 0 to 1 and vice-versa). Suppose that initially, $\ell(v) = 0$ for all $v \in V$. We show that there exists a set $S \subseteq V$ such that applying $T(v), v \in S$ in any order makes $\ell(v) = 1$ for $v \in V$. This shows that we can turn on all the lights in Imperial Hall.

Observe first that applying $T(v)$ and then $T(w)$ achieves the same effect as applying $T(w)$ and then $T(v)$ i.e. the order of application of the transformations does not matter. Let A be the 0-1 adjacency matrix of G i.e. let $A(v, w) = 1$ iff $w \in N(v)$. In addition put $A(v, v) = 1$ for $v \in V$. The set of transformations corresponding to S will turn on all of the lights iff $A\mathbf{1}_S = \mathbf{1}_V$ where $\mathbf{1}_S$ is the 0-1 vector indexed by V such that there is a 1 in component v iff $v \in S$.

Our claim amounts to saying that there exists S such that $A\mathbf{1}_S = \mathbf{1}_V$ where calculations are done in the binary field. If there is no such $\mathbf{1}_S$ then basic linear algebra theory tells us that there exists x such that $x^T A = 0$ and $x^T 1_V \neq 0$.

Since A is symmetric, this means that $Ax = 0$ as well. Let $x = 1_S$. Then S has the following properties:

(a) $|S \cap N(v)|$ is odd for all $v \in V$. This is a consequence of $Ax = 0$.

(b) $|S|$ is odd. This is a consequence of $x^T 1_V \neq 0$.

Now consider the sub-graph of G induced by S. Every vertex has odd degree by (a). But in any graph, the number of odd vertices is even. Contradiction.

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References

[1] K. Sutner, Linear Cellular Automata and the Garden of Eden, Mathematical Intelligencer 11 (1989) 49-53.