Hiking: In the Padurea forest there are 100 rest stops. There are 1000 trails, each connecting a pair of rest stops. Each trail e has a level of difficulty $\ell(e)$. No two trails have the same difficulty. An intrepid hiker, Sendeirismo has decided to spend a vacation by taking a hike consisting of 20 trails of ever increasing difficulty. Can he be sure that it can be done?
He is free to choose the starting rest stop and the 20 trails form a sequence where the start of one trail is the end of a previous one.
Solution: We generalise the problem as follows: Let $G = (V, E)$ be a graph with $|V| = n, |E| = m$ and suppose that $e_1 < e_2 < \cdots < e_m$ is an arbitrary ordering of the edges E. Let $d = 2m/n$ be the average degree. There is a trail $f_1, f_2, \ldots, f_k$ such that $k \geq d$ and $f_1 < f_2 < \cdots < f_k$.
Proof: Place n walkers, one at each vertex. Then go through the edges of E one by one and when you come to edge $e = (u, v)$ the current walkers at u, v change places. Let P_v be the trail that is traversed by the walker originally at vertex v. Then each P_v follows an increasing sequence of edges. Now

$$\sum_{v \in V} |P_v| = 2m$$

and so there exists v such that $|P_v| \geq 2m/n$.
This puzzle and its beautiful proof was communicated to us by Peter Winkler.